YEARS of MATHEMATICS TEACHER

from the 1990s

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Climbing Around on the Tree of Mathematics

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Dan Kennedy's award-winning article appeared in 1995. What was the familiar scenery for mathematics teachers who journeyed to Kennedy's metaphoric forest just over a decade ago? NCTM had published its first Standards document, *Curriculum and Evaluation Standards for School Mathematics,* in 1989, and *Assessment Standards for School Mathematics* (1995) was hot off the press. The graphing calculator was no longer a brand-new tool: In many schools, teachers used them side-by-side with computer software.

Kennedy argues that the graphing calculator allows his students to discover the beauty and power of mathematics regardless of their ability to FOIL or factor. That students *should* experience the beauty of mathematics was a goal shared by many of our predecessors, though they lacked the technology that Kennedy could call on. The 1923 report *The Reorganization of Mathematics in Secondary Education*—a document as widely quoted in its day as the Standards are today—presented a list of the cultural aims of mathematics instruction, including—

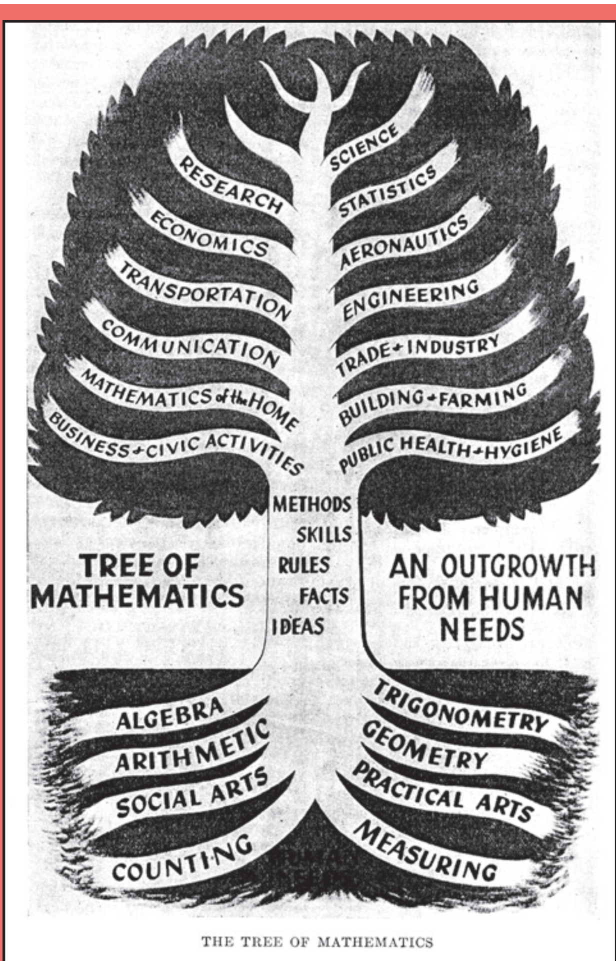
- appreciation of beauty in the geometrical forms of nature, art, and industry;
- ideals of perfection as to logical structure, precision of statement and of thought; and
- appreciation of the power of mathematics and the role that mathematics and abstract thinking have played in the development of civilization, in particular in the sciences, industry, and philosophy

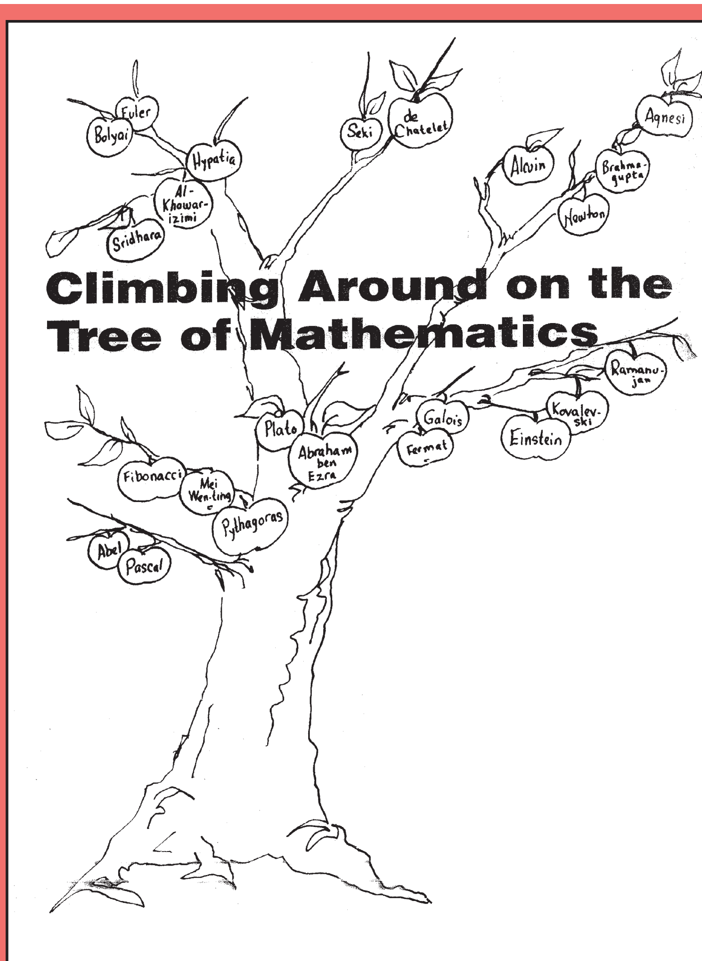
Kennedy's concluding advice is to revise the curriculum—rake away the dead leaves. He offers no specifics, but readers of the *Mathematics Teacher* know the pages of the journal have detailed countless suggestions for improving the curriculum.

When I was in graduate school umpteen years ago, I was occasionally visited by anxiety attacks centered on such thoughts as "What am I doing here?" Specifically, I wondered how someone as obviously inconsequential as myself could hope to contribute any original thought to the vast lexicon of original thoughts known collectively as Mathematics. Without that obvious prerequisite, what would I use as a dissertation?

One afternoon, while I was suffering such an attack in the office of my thesis advisor, he consoled me by suggesting that the entire body of Mathematical Knowledge was very much like a tree. The main body was this big trunk of general knowledge, from which protruded different branches of concentration, from which emerged smaller branches of specialization, from which finally sprouted twigs of truly arcane trivia. All that I had to do to expand the tree was to ascend the trunk, climb out on a branch, crawl along some branchlets to reach some twig, then reach out and extend that *one little twig* by some tiny amount. Doctoral dissertations, in other words, were not about branches; they were about twigs.

Encouraged by this clarification of my mission, I returned to my studies with renewed optimism. Eventually I climbed the trunk to the point where I could access the branch of Combinatorics. From there I shinnied out to the smaller branch of Combinatorial Geometries, found a twig called





Original "Tree of Mathematics" artwork from ***Mathematics Teacher*** 48 (May 1948) and the artwork that accompanied this article, ***Mathematics Teacher*** 88 (September 1995)

Factorizations of Combinatorial Geometries, and tentatively squeezed forth a twiglet called Majors of Factorizations of Combinatorial Geometries. That twiglet might eventually bear some kind of fruit, but I will not be there to see it; I long ago retreated back to the safety of the trunk, and here I am—a high school teacher.

Readers must admit that this description is a remarkably accurate portrayal of how the body of mathematics *grows.* Still, we have hardly begun to explore the richness of the tree metaphor if we limit ourselves to growth. In fact, this description is a remarkably accurate portrayal of how the body of mathematics *works.* The researchers who are recognized as doing the serious and important mathematics are laboring at the ends of branches, whereas those of us who aspire to teach undergraduates are coaxing our students up the trunk, praying that someday a few of them might be inspired to climb past us on their way to exploring the richness of the foliage beyond. The fact that the trunk has not changed perceptibly in centuries of growth does not concern us, nor do the trunk's unfortunate characteristics of being hard, rigid, unyielding, monotonous, and increasingly far removed from the beauty at the end of the branches. Why should we mathematicians, generally respected for our intelligence and perception, fail to be concerned about these things? It is because we realize that access to the branches is allowed *only* by way of the trunk—because the trunk is the foundation of the tree—and the safest path up that trunk is the same path along which we ourselves climbed decades ago.

If that analogy makes sense to you, and it certainly should if you have devoted your life to teaching algebra, then let me remind you that it makes no sense at all to the millions of educated people who have decided, most of them since high school, that they have no use for mathematics. They tried to climb our tree, but they just could not get their hands around that enormous, intimidating trunk. Do not worry about them, though; they went on to discover other trees in the forest. I am sure you have noticed that in the branches of those other trees, many of these climbers are a lot closer to the sun than we are. They can see for miles in many directions, but, ironically, they still do not know much about our stately and imposing Tree of Mathematics. They know even less about what we are doing in there, huddled by the trunk, in the darkness cast by the thick, obscuring branches. Luckily, they assume that we are doing something important. It is, after all, a magnificent tree, and everyone who gazes at its inscrutable glory hopes that someday, somehow, he or she will raise a child who can climb it.

Now before I give the impression that I think mathematics teachers spend their lives in the dark, let me remind everyone that I am a mathematics teacher myself. Most of my best friends are mathematics teachers. Also, let me acknowledge that every reader can probably point with fondness to a mathematics teacher in the past who has made a difference in his or her life. However, I dare say that this fondness will arise because the teacher taught you about studying, or perseverance, or believing in yourself, or some such enduring lesson of human existence; it will probably not be because that teacher taught you how to rationalize the denominator or how to factor a trinomial—even though that is what the two of you spent most of your time together doing. You were climbing that trunk, just as everyone else around you was struggling to do, but because you climbed it while looking up at your teacher, you managed to catch a few glimpses of the sky beyond.

The problem is, not everyone on that trunk was looking up. Some were too scared; some became convinced that their arms were simply too short to hug that trunk; still others became discouraged every time they saw how far away they were from the foliage that was to be their goal. Perhaps they could not look up; after all, we, as teachers, did focus most of their attention on the finding of roots! Whatever the situation, we were scaring away many creative minds, some of whom have since gotten back at us by portraying us negatively in teen-oriented movies. Moreover, we were not getting many of our climbers very far up that tree. I am not here to blame the teachers for this lack of success, though; it was definitely not our fault, which is why I am writing about trees.

So let us leave the tedious trunk for a while and talk about the situation farther up the tree, where things are not much better. There, you will recall, everyone is off on a different branch specializing in one particular twig, virtually unaware of what is happening on the branches elsewhere in the tree. This specialization has created another interesting public relations problem for mathematicians. I am sure that you all remember reading about the apparent proof of Fermat's Last Theorem, probably the most exciting news story in our lifetime concerning real mathematics. This event was to be a *very* big twig, and the tree was quivering with excitement. The story even made the *New York Times*—twice. But even while being quoted for the record, professional mathematicians acknowledged that only a handful of experts would be able to understand the proof, since, essentially, nobody else was far enough out on that particular branch of the tree. In other words, mathematicians could not explain to reporters the biggest result in their own subject in this century. Fortunately the reporters were accustomed to this lack of clarification, since they spend much of their time dealing with politicians.

This last example, I think, finally illustrates the real problem that we all face in mathematics education today. What has happened is that the tree of mathematics has grown to the point at which it is much too big to know. (Indeed, so have all the other trees in the educational forest, especially the history tree, which grows in real time. But that is another story.) You can know a lot about a branch and everything about a twig, but nobody can know the entire tree—and we know enough about mathematics to realize that. We forgave ourselves long ago for not *knowing* all the mathematics, realizing that it would not affect our ability to *appreciate, use,* and *do* mathematics. As mathematicians we must *be* specialists, but we still *teach* generalists. Unable to teach them about the whole tree, we choose to teach them about the safest part of the tree that we know: that sturdy, immutable trunk, which will at least give them the foundation they need for getting up into the branches—if they can survive the climb.

Our choice has also fostered a certain style of teaching in many of us, the style that seeks to cover the necessary material as efficiently as possible, namely the "Here's how you do it. Any questions? Good. Do it" style of teaching. Unless you expose them to the part of the trunk in your lesson plan for the day, you will never get through the syllabus.

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1990s

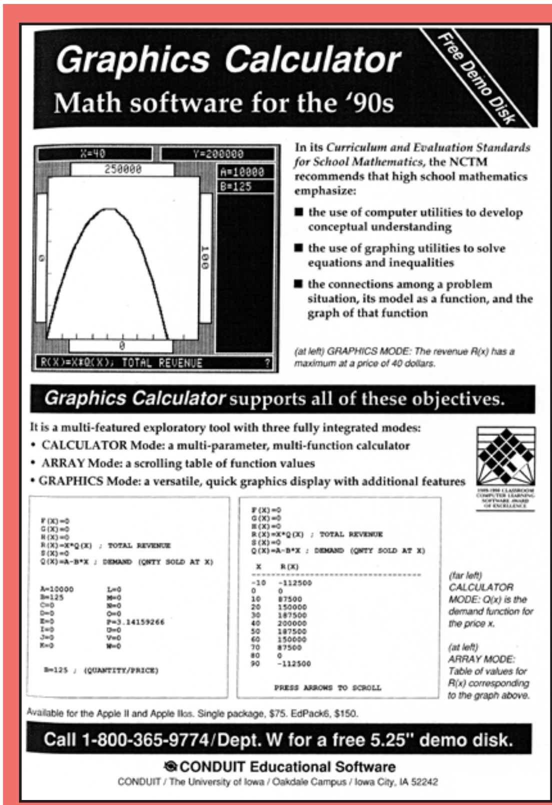
1991 *Professional Standards for Teaching Mathematics* is published.

1992 Format of the journal changes from 6.75 × 10 inches to its current size, 8.375 × 11 inches.

1994 The Third International Mathematics and Science Study (TIMSS) is conducted in more than forty countries.

1995 NCTM issues *Assessment Standards for School Mathematics.*

Andrew Wiles proves Fermat's Last Theorem.



Sample ad, *Mathematics Teacher* 83 (April 1990)

You have so much to cover and so little time. As the tree has grown bigger and bigger, the textbooks have simply grown right along with it, until now we have those seventy-five-dollar, hernia-producing behemoths that are so ridiculously impossible to cover that nobody even tries anymore. *We* realize that the course is inside the textbook somewhere, and we can guide our students through it if we have enough experience on the trunk of the tree, but what do the *students* think when they see that book? Would you buy a toaster oven if the owner's manual was 600 pages long? Of course not! You would much rather give up toast.

If one good thing can be said about the tree's getting so enormous, it is this: We can finally begin to let go of the idea that some significant subset of the tree exists that every educated human being, past, present, and future, should know. This idea will not die easily, to be sure, but I do think that it is useful to question that time-honored assumption. Take, for example, the quadratic formula. I watched Johnny Carson quote that formula from memory during his monologue one evening, to thunderous applause from an audience of apparent nonmathematicians who recognized it immediately as humorous. He went on to say that he had remembered that formula from high school in Nebraska and added that his teacher had promised him that he and his classmates would use it later in life. That rash prediction drew a laugh from the audience, but only because they all knew what was coming. With his usual impeccable timing, he rode the swell of that first laugh to its conclusion, then pointed out that he had waited fifty years before finally using that formula for the first time—to get a laugh in his monologue.

I will not ask if you have been forced to make similar promises to your students over the years, but I would be surprised if you have lasted long in this business without doing so. Just think of how much of your course, whatever it is, is predicated on the assumption that you are preparing your students for future mathematics courses. That assumption is what teaching on the trunk of the tree is all about. First-year algebra leads to geometry, which leads to second-year algebra, which leads to precalculus, which leads to calculus, which for most students has historically led to the exit. We essentially spend twelve years getting our students ready for calculus, and when they get there, they discover that it is 300 years old, filled with the same calculations they hated in high school, and not exactly worth twelve years of anticipation. So they shinny down the mathematics tree and strike out into the forest, armed at least with those twelve rich years of valuable mathematical learning: trigonometry identities, the rational-root theorem, synthetic division, side-angle-side, FOIL, the commutative property of addition—hey, you name it. Then, on the first day on the job out in the real world, someone notices that they have twelve years of mathematics on their transcript and says with relief, "At last, someone who knows some math! Come here and explain this spreadsheet to me." What will their twelve years do for them then?

I will confess to having fabricated that previous scenario for dramatic rhetorical effect rather than as a reflective argument for revolutionary change. I am not yet inclined to let my students graduate without having studied the quadratic formula. I happen to think that good reasons for teaching it exist, but one is not because my students will use it later in life. It is, after all, part of the trunk, and I do not want my students to hang around the trunk forever. I want them up in the tree. Moreover, some other things can be found in the trunk about which I am not so fond, such as rationalizing the denominator, and I no longer feel guilty if my students can climb the tree without seeing those. Can that climb be made? Can students access the tree without climbing up the trunk? The interesting thing, the miraculous thing, the thing that has changed my view of teaching forever, is that yes, now they actually can.

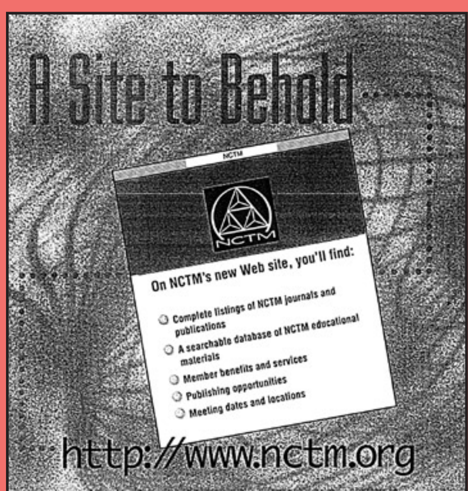
Look around you in the tree of mathematics today, and you will see some new youngsters playing around in the branches. They are exploring parts of the tree that have not seen this kind of action in centuries, and they did not even climb the trunk to get there. Do you know how they did it? They cheated: they used a ladder. They climbed directly into the branches using a prosthetic extension of their brains known in the education business as technology. They got up there with graphing calculators. You can argue all you want about whether they deserve to be up so high, and about whether they might fall, but that argument will not change the fact that they are there, straddled alongside the best trunk-climbers in the tree—and most of them are glad to be in that tree. Now I ask you: Is that beautiful, or is that bad? Let me warn you that your answer to that beguiling question will probably affect the way you teach for the rest of your lives.

For the record, I think that it is beautiful that students of all ages and abilities can access the higher branches of the tree of mathematics without having to struggle up the trunk. I also think that it is healthier for the tree and, ultimately, for the whole educational forest. That is why I plan to spend the rest of my career as a teacher steadying ladders for my students and watching them solve meaningful problems up in the branches. If some of my students miss part of the trunk or, perish the thought, know less about finding roots, then so be it. Remember: *The tree is too big to know anyway*—and I want my students to enjoy the view.

The graphing calculator changed my entire approach to teaching. The first thing I did was let them use it—all the time. I could then focus on how I would get the students to use it, which in turn encouraged me to focus on students' learning rather than on my own teaching. I saw how well they worked with each other with the calculators, so I began to develop ways to make them work together to discover the mathematics. I now start each class by having them work together on a problem, often the type of problem that I formerly used in a lecture to involve student interest in the lesson of the day—only now I wait for *them* to *discover* the lesson of the day. Once I saw that they could actually make that discovery, I realized how useless my crisp set of lecture notes had been all those years. Now there is no turning back.

The technology that has made the difference in the tree is, of course, computer technology, but it would never have revolutionized the classroom experience were it not for its availability in these small, remarkably inexpensive packages. We call this package a graphing calculator, but it is actually a computer—a computer with a very focused mission, running sophisticated internal software that is devoted to mathematics. It does simple mathematics for those with simple tastes, and it does advanced mathematics for those with advanced tastes. More significantly, it also does *advanced* mathematics for those with *simple* tastes. A chimpanzee, for example, can produce a perfect graph of $y = \sin x$ while clapping his feet with excitement. Most would argue that the chimp will not understand what it has, and I agree, but some would argue that first-year-algebra students would not understand what they have either, and I disagree. Not only can first-year-algebra students understand that it is a function, but they can understand that it is bounded, periodic, continuous, sometimes increasing and sometimes decreasing, with a maximum of 1 and a minimum of –1. They can also understand that the graph changes curvature every time it crosses the x-axis and, with a little explanation, they can probably even appreciate that it models harmonic motion. Can they recognize that waves look like that? Of course they can, and if you have an oscilliscope, you can prove it to them. Remarkably, they will be able to understand all that without knowing anything about opposite-over-hypotenuse, the unit circle, reference angles, or a radian. They can learn all sorts of things about $y = \sin x$ by just playing around on the tree of mathematics.

One of my advisees, not a student of higher mathematics, asked me recently what he could graph on his brand new TI-82 calculator to make a neat picture. I told him to put it into POLAR mode and graph $r = \sin 6\theta$. He liked that so much that he tried $\sin 66\theta$. Aren't these great pictures? You do not have to know a lot of mathematics to appreciate these graphs, and I will bet that students who do see these graphs will have greater respect for polar graphs and trigonometry when they encounter them again farther up the tree. We also graphed $r = \sin 666\theta$, which simply duplicated the graph of $r = \sin 6\theta$. To appreciate that graph, you have to know some mathematics!



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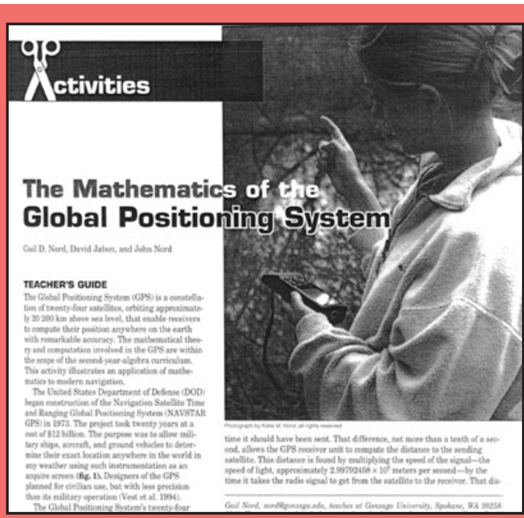
In closing, lest anybody accuse me of not seeing the forest for the trees, let me overwork this arboreal metaphor one more time by applying it to the traditional United States curriculum. Our educational forest is very much like the majestic maple forests of my Algonquin summer home. A maple forest needs centuries to develop, but once its trees are in place, the maples will dominate the landscape forever. Why? Because maple trees drop their leaves every fall, and those leaves eventually form a dense carpet over the forest floor, keeping all but the strongest seedlings from reaching the life-giving soil below. The maples then produce millions of seeds, and theirs are the only seedlings with the strength to pierce the carpet. Maple forests, in other words, have inadvertently evolved a perfect strategy for producing clones of themselves forever.

All the trees in our educational forest are bearing some strong and healthy seedlings. Many of our students leave us and become fine, productive citizens: scientists, teachers, authors, philosophers, doctors, lawyers, mothers, fathers, and even mathematicians. But while our stately academic trees are blooming high above, you might have noticed that not much is happening below to regenerate the forest itself. Look around you: The forest floor is littered with the dead leaves of centuries of curricular material, forming a dense and impenetrable mat through which only the strongest of young scholars can pierce. Many of those leaves came from the tree of mathematics, although the other academic disciplines have certainly contributed their share. Even after the branches of active mathematics have sloughed them off, we keep our leaves around out of respect, or out of tradition, or because they are still in the textbook, or because we are terrified that some teacher in some future course will assume that our students know them and they will not. Although such debris is only a side effect of how trees grow, nothing of deliberately malicious design could ever have more effectively kept new trees out of the forest than that litter on the forest floor. The time our students spend with us being educated is very precious; we should not be wasting any of it.

Ironically, most good schools encourage all students to take mathematics every year, precisely because they see the aching need for mathematical understanding to cope with our increasingly technological society. Little do they realize that we are teaching them the same classical results that we felt their great-grandparents needed to cope with the industrial revolution. When do we teach them about the technology that will make the technological society technological? When will they learn what these machines and bigger computers can do? Our curriculum already contains far too much material to cover, and the dead leaves just keep


Activities

The Mathematics of the Global Positioning System

Gail D. Nord, David Jabon, and John Nord

TEACHER'S GUIDE

The Global Positioning System (GPS) is a constellation of twenty-four satellites, orbiting approximately 20 200 km above sea level, that enable receivers to compute their position anywhere on the earth with remarkable accuracy. The mathematical theory and computation involved in the GPS are within the scope of the second-year-algebra curriculum. This activity illustrates an application of mathematics to modern navigation.

The United States Department of Defense (DOD) began construction of the Navigation Satellite Time and Ranging Global Positioning System (NAVSTAR GPS) in 1973. The project took twenty years at a cost of $12 billion. The purpose was to allow military ships, aircraft, and ground vehicles to determine their exact location anywhere in the world in any weather using such instrumentation as an acquire screen (**fig. 1**). Designers of the GPS planned for civilian use, but with less precision than its military operation (Vest et al. 1994).

The Global Positioning System's twenty-four

time it should have been sent. That difference, not more than a tenth of a second, allows the GPS receiver unit to compute the distance to the sending satellite. This distance is found by multiplying the speed of the signal—the speed of light, approximately 2.99792458×10^8 meters per second—by the time it takes the radio signal to get from the satellite to the receiver. That dis-

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accumulating. If the educational forest is ever to be transformed, then I submit that the decay on the floor is the next frontier.

Now that the ladder of technology, in our example of the graphing calculator, has demonstrated its effectiveness in getting new students into the trees in their quest for sunlight, I doubt that the forest will ever be the same. Soon everyone will be buzzing about electronic classrooms, cross-disciplinary learning, multicultural studies, information superhighways, and networking—curricular concerns that do not fit neatly into the current educational forest. I see them as new holes in the forest canopy that afford wonderful growth opportunities, if only some new trees could take root to take advantage of them. Can we expect some new trees in our educational forest in the near future? Well, nothing is stopping them now but the dead leaves of the way we were. The ladder has served us well; now we must bring on the rake.

Not long ago, I attended a meeting hosted by the College Board, at which thirty members of the professional mathematics community gathered to advise the Advanced Placement Calculus Committee on how the AP curriculum should be reformed to conform to the best calculus courses now being offered in our colleges and universities. They did not always agree, but one thing was for certain: these people came with rakes! The AP committee will now spend several months drafting a new course description for a leaner, livelier calculus that sometime around 1997 will officially become the AP course we teach. If you want to get a head start, just get into the branches and away from the trunk.

My students and I will see you there.

The author would like to thank Doug Kelly as a source for the tree metaphor. ∞