Worksheet on Ring Isomorphisms

Let $L$ be the ring defined in Problem 22 in Section 3.1 on page 53. So $L$ is the set of positive real numbers and we define a new addition and multiplication on $L$ by

$$a \oplus b = ab \quad \text{and} \quad a \otimes b = a^\log b$$

where $\log b$ is the natural log of $b$.

We want to show that $L$ is isomorphic to $\mathbb{R}$ as rings. Our first task is to create a homomorphism from $L$ to $\mathbb{R}$ that is one-to-one and onto. We have a couple of hints on how to do this.

In Problem 22 of Section 3.1, we saw $0_L = 1$. If $\phi : L \to \mathbb{R}$ is an isomorphism of rings, what must $\phi(1)$ be? (Hint: See Theorem 3.12.)

We also saw that $L$ has an identity, which was $e$. Since $\phi$ is an isomorphism, we can use Theorem 3.12 to determine what $\phi(e)$ must be. What is $\phi(e)$?

Use the properties of logs and homomorphisms to determine $\phi(e^2), \phi(e^3)$, and $\phi(e^{-1})$.

Make a guess as what might be an appropriate function for $\phi$.

Now prove that the function $\phi$ you defined is an isomorphism of rings.