Krzys' Ostaszewski: http://www.krzysio.net
Author of the "Been There Done That!" manual for Course P/1

Instructor for Course P/1 seminar: http://www.math.ilstu.edu/actuary/prepcourses.html
If you find these exercises valuable, please consider buying the manual or attending the seminar, and if you can't, please consider making a donation to the Actuarial Program at Illinois State University: https://www.math.ilstu.edu/actuary/giving/
Donations will be used for scholarships for actuarial students. Donations are tax-deductible to the extent allowed by law.

Questions about these exercises? E-mail: krzysio@krzysio.net

Dr. Ostaszewski's online exercise posted January 21, 2006
Compute the expected value of the random number of coin tosses until a run of k successive heads occurs when the tosses are independent and each lands on heads with a probability $\frac{1}{2}$.

A. 2^{k-1} B. 2^k+2 C. $2^{k+1}-2$ D. 2^{k+1} E. Does not exist

Solution.
Let X be the random number of tosses till k successive heads occur. Note that if $X=n$, then $n \geq k$ and the last k tosses were heads, the one just before those k was not heads (i.e., it was tails), and there were no k consecutive heads on the $n-k-1$ rolls before that. Therefore,

$$\Pr(X=n) = \underbrace{\left(\frac{1}{2}\right)^k}_{\substack{\text{Heads on the}\\ \text{last } k \text{ tosses}}} \cdot \underbrace{\frac{1}{2}}_{\substack{\text{Tails on the toss}\\ \text{just before that}}} \cdot \underbrace{\Pr(X>n-k-1)}_{\substack{\text{No } k \text{ consecutive heads}\\ \text{on the first } n-k-1 \text{ tosses}}} = \left(\frac{1}{2}\right)^{k+1} \cdot \underbrace{\Pr(X \geq n-k)}_{=\Pr(X>n-k-1) \text{ because } X \text{ is discrete}}.$$

This is equivalent to the statement that for $n \geq k$, $\Pr(X \geq n-k) = 2^{k+1} \cdot \Pr(X=n)$. We also have $X \geq k$ with probability 1 (and, notably, also $X \geq 0$ with probability 1, so that we can use the Darth Vader Rule) and $\Pr(X=k) = \left(\frac{1}{2}\right)^k = \frac{1}{2^k}$, while

$$\Pr(X=k+1) = \underbrace{\frac{1}{2}}_{\text{Tails on first toss}} \cdot \underbrace{\left(\frac{1}{2}\right)^k}_{\text{Heads on 2nd, 3rd, ..., and } (k+1)\text{-st toss}} = \frac{1}{2^{k+1}},$$

$$\Pr(X=k+2) = \underbrace{1}_{\text{Anything on the first toss}} \cdot \underbrace{\frac{1}{2}}_{\text{Tails on second toss}} \cdot \underbrace{\left(\frac{1}{2}\right)^k}_{\text{Heads on 3rd, 4th, .., and } (k+2)\text{-nd toss}} = \frac{1}{2^{k+1}},$$

and the same way, as long as $1 \leq r \leq k$,

$$\Pr(X=k+r) = \underbrace{1 \cdot 1 \cdot \ldots \cdot}_{\text{Anything on first } r-1 \text{ tosses}} \underbrace{\frac{1}{2}}_{\text{Tails on } r-\text{th toss}} \cdot \underbrace{\left(\frac{1}{2}\right)^k}_{\text{Heads on } (r+1)\text{-st, .., and } (r+k)\text{-th toss}} = \frac{1}{2^{k+1}}.$$

The above gives us probabilities that X attains the values of 1, 2, 3, ..., k, $k + 1$, ..., $2k$, and beyond those values we can use the recursive formula we derived in the first step. Therefore,

$$E(X)=\sum_{n=1}^{+\infty}\Pr(X\geq n)=\sum_{n=1}^{k-1}\Pr(X\geq n)+\sum_{n=k}^{+\infty}\Pr(X\geq n)=\sum_{n=1}^{k-1}1+\sum_{n=k}^{+\infty}\Pr(X\geq n)=$$

$$=(k-1)+\sum_{n=2k}^{+\infty}\Pr(X\geq n-k)=(k-1)+\sum_{n=2k}^{+\infty}2^{k+1}\cdot\Pr(X=n)=$$

$$=(k-1)+2^{k+1}\cdot\sum_{n=2k}^{+\infty}\Pr(X=n)=(k-1)+2^{k+1}\cdot\left(1-\sum_{n=0}^{n=k-1}\underbrace{\Pr(X=n)}_{=0}-\sum_{n=k}^{n=2k-1}\Pr(X=n)\right)=$$

$$=(k-1)+2^{k+1}\cdot\left(1-\frac{1}{2^k}-\frac{1}{2^{k+1}}-\ldots-\frac{1}{2^{k+1}}\right)=(k-1)+2^{k+1}\cdot\left(1-\frac{1}{2^k}-\frac{k-1}{2^{k+1}}\right)=2^{k+1}-2.$$

Answer C.