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Exercise for October 22, 2005
You are given that the joint PDF of random variables X and Y is

$$f_{X,Y}(x,y) = x + y$$

for $0 < x < 1,\ 0 < y < 1,$ and 0 otherwise. Find $\mathrm{Var}(X - Y)$.

A. 1 B. $\frac{1}{2}$ C. $\frac{1}{3}$ D. $\frac{1}{4}$ E. $\frac{1}{6}$

Solution.
Note that

$$\int_0^1 \int_0^1 (x+y)\,dxdy = \int_0^1 \left(\frac{1}{2}x^2 + xy \right)\Big|_{x=0}^{x=1} dy = \int_0^1 \left(\frac{1}{2} + y \right) dy = \left(\frac{1}{2}y + \frac{1}{2}y^2 \right)\Big|_{y=0}^{y=1} = 1.$$

We have

$$f_X(x) = \int_0^1 (x+y)\,dy = \left(xy + \frac{1}{2}y^2 \right)\Big|_{y=0}^{y=1} = x + \frac{1}{2},$$

for $0 < x < 1,$ and 0 otherwise. Because of symmetry, the PDF of Y is the same.
Therefore,

$$E(X) = \int_0^1 x \cdot \left(x + \frac{1}{2} \right) dx = \left(\frac{1}{3}x^3 + \frac{1}{4}x^2 \right)\Big|_{x=0}^{x=1} = \frac{1}{3} + \frac{1}{4} = \frac{4}{12} + \frac{3}{12} = \frac{7}{12},$$

$$E(Y) = E(X) = \frac{7}{12},$$

$$E\left(X^2\right) = \int_0^1 x^2 \cdot \left(x + \frac{1}{2} \right) dx = \left(\frac{1}{4}x^4 + \frac{1}{6}x^3 \right)\Big|_{x=0}^{x=1} = \frac{1}{4} + \frac{1}{6} = \frac{3}{12} + \frac{2}{12} = \frac{5}{12},$$

$$E\left(Y^2\right) = E\left(X^2\right) = \frac{5}{12},$$

and the expected value of XY is

$$E(XY) = \int_0^1 \int_0^1 xy(x+y)\,dxdy = \int_0^1 \int_0^1 \left(x^2y + xy^2\right) dxdy = \int_0^1 \left(\frac{1}{3}x^3y + \frac{1}{2}x^2y^2 \right)\Big|_{x=0}^{x=1} dy =$$

$$= \int_0^1 \left(\frac{1}{3}y + \frac{1}{2}y^2 \right) dy = \left(\frac{1}{6}y^2 + \frac{1}{6}y^3 \right)\Big|_{x=0}^{x=1} = \frac{1}{3}.$$

This gives

$$\begin{aligned}
\mathrm{Var}(X-Y) &= \mathrm{Var}(X)+\mathrm{Var}(Y)-2\mathrm{Cov}(X,Y)= \\
&= \left(E\left(X^2\right)-\left(E(X)\right)^2\right)+\left(E\left(Y^2\right)-\left(E(Y)\right)^2\right)-2\cdot\left(E(XY)-\left(E(X)\right)\cdot\left(E(Y)\right)\right)= \\
&= \left(\frac{5}{12}-\frac{49}{144}\right)+\left(\frac{5}{12}-\frac{49}{144}\right)-2\cdot\left(\frac{1}{3}-\frac{49}{144}\right)= \\
&= \left(\frac{60}{144}-\frac{49}{144}\right)+\left(\frac{60}{144}-\frac{49}{144}\right)-2\cdot\left(\frac{48}{144}-\frac{49}{144}\right)=\frac{24}{144}=\frac{1}{6}.
\end{aligned}$$

Answer E.