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If you find these exercises valuable, please consider buying the manual or attending the seminar, and if you can't, please consider making a donation to the Actuarial Program at Illinois State University: https://www.math.ilstu.edu/actuary/giving/
Donations will be used for scholarships for actuarial students. Donations are tax-deductible to the extent allowed by law.
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**Society of Actuaries November 2005 Course M Examination, Problem No. 17, and Dr. Ostaszewski's online exercise posted December 24, 2005**

The length of time, in years, that a person will remember an actuarial statistic is modeled by an exponential distribution with mean $\frac{1}{Y}$. In a certain population, $Y$ has a gamma distribution with $\alpha = 2$ and $\beta = \frac{1}{2}$. Calculate the probability that a person drawn at random from this population will remember an actuarial statistic less than $\frac{1}{2}$ year.

A. 0.125 B. 0.250 C. 0.500 D. 0.750 E. 0.875

Solution.

Let $T$ be the random time for the length of which a person remembers an actuarial statistic. We are told that $f_T\left(t \middle| Y = y\right) = ye^{-yt}$, as well as $s_T\left(t \middle| Y = y\right) = e^{-yt}$. Note the following

$$\Pr\left(T \geq \frac{1}{2}\right) = \int_{0.5}^{+\infty} f_T(t)\,dt = \int_{0.5}^{+\infty}\left(\int_{0}^{+\infty} f_{T,Y}(t,y)\,dy\right)dt = \int_{0}^{+\infty}\int_{0.5}^{+\infty} f_{T,Y}(t,y)\,dt\,dy =$$

$$= \int_{0}^{+\infty}\underbrace{\left(\int_{0.5}^{+\infty} f_T\left(t \middle| Y = y\right)dt\right)}_{s_T(0.5|Y=y)=e^{-0.5y}}\cdot f_Y(y)\,dy = \int_{0}^{+\infty} e^{-0.5y}\cdot f_Y(y)\,dy.$$

Recall that the MGF of a gamma random variable $Y$ with parameters $\alpha$ and $\beta$ is $M_Y(u) = \left(\frac{\beta}{\beta - u}\right)^{\alpha}$. Therefore

$$\Pr\left(T \geq \frac{1}{2}\right) = \int_{0}^{+\infty} e^{-0.5y}\cdot f_Y(y)\,dy = M_Y(-0.5) = \left(\frac{0.5}{0.5-(-0.5)}\right)^2 = \frac{1}{4}.$$

This gives

$$\Pr\left(T < \frac{1}{2}\right) = 1 - \Pr\left(T \geq \frac{1}{2}\right) = \frac{3}{4}.$$

Answer D.