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Exercise for April 1, 2006

November 1981 Course 110 Examination, Problem No. 49
Suppose $X_j = Z_j - Z_{j-1}$, where j = 1, 2, …, n and $Z_0, Z_1, Z_2, ..., Z_n$ are independent and identically distributed with common variance σ^2. What is $\text{Var}\left(\frac{1}{n}\sum_{j=1}^{n} X_j\right)$?

A. $\frac{\sigma^2}{n}$ B. $n\sigma^2$ C. $\frac{\sigma}{\sqrt{n}}$ D. $\frac{2\sigma^2}{n^2}$ E. $\frac{2\sigma^2}{n}$

Solution.
Since any Z_j and Z_k are independent,

$$\text{Var}\left(Z_j - Z_k\right) = \sigma^2 + \sigma^2 = 2\sigma^2.$$

Therefore,

$$\text{Var}\left(\frac{1}{n}\sum_{j=1}^{n} X_j\right) = \frac{1}{n^2}\text{Var}\left(\left(Z_1 - Z_0\right) + \left(Z_2 - Z_1\right) + \ldots + \left(Z_n - Z_{n-1}\right)\right) =$$

$$= \frac{1}{n^2}\text{Var}\left(Z_n - Z_0\right) = \frac{2\sigma^2}{n^2}.$$

Answer D.