Krzys' Ostaszewski: http://www.math.ilstu.edu/krzysio/
Author of the BTDT Manual for Course P
available at http://smartURL.it/krzysioP or http://smartURL.it/krzysioPe
Instructor for online Course P/1 seminar: http://smartURL.it/onlineactuary
Exercise for April 8, 2006

X and Y have a bivariate normal distribution with $E(X)=0,\ E(Y)=-1,$ and $E(XY)=1.$ If $E(Y|X=2)=1,$ and $E(X|Y=0)=\frac{1}{16},$ find $\mathrm{Var}(Y|X=-2).$

A. 3 B. 4 C. 9 D. 12 E. 15

Solution.
Let us write $E(X)=\mu_X,\ E(Y)=\mu_Y,\ \mathrm{Var}(X)=\sigma_X^2,\ \mathrm{Var}(Y)=\sigma_Y^2,$ and

$$\rho=\rho_{YX}=\frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\cdot\mathrm{Var}(Y)}}=\frac{E(XY)-E(X)\cdot E(Y)}{\sigma_X\cdot\sigma_Y}.$$

Note that

$$\rho=\frac{1-0\cdot(-1)}{\sigma_X\cdot\sigma_Y}=\frac{1}{\sigma_X\cdot\sigma_Y}.$$

Recall that

$$E(Y|X=x)=\mu_Y+\rho\sigma_Y\cdot\frac{x-\mu_X}{\sigma_X}.$$

Therefore,

$$1=E(Y|X=2)=-1+\frac{1}{\sigma_X\cdot\sigma_Y}\cdot\sigma_Y\cdot\frac{2}{\sigma_X}=-1+\frac{2}{\sigma_X^2},$$

so that $\sigma_X=1.$ Furthermore,

$$E(X|Y=y)=\mu_X+\rho\sigma_X\cdot\frac{y-\mu_Y}{\sigma_Y}.$$

Hence,

$$\frac{1}{16}=E(X|Y=0)=0+\frac{1}{\sigma_X\cdot\sigma_Y}\sigma_X\cdot\frac{0-(-1)}{\sigma_Y}=\frac{1}{\sigma_Y^2},$$

and $\sigma_Y=4.$ This implies that $\rho=\frac{1}{\sigma_X\cdot\sigma_Y}=\frac{1}{4}.$ Finally,

$$\mathrm{Var}(Y|X=-2)=(1-\rho^2)\sigma_Y^2=\left(1-\frac{1}{16}\right)\cdot16=15.$$

Answer E.