Krzys' Ostaszewski http://www.math.ilstu.edu/krzysio/
Author of the BTDT Manual for Course P/1
available at http://smartURL.it/krzysioP or http://smartURL.it/krzysioPe
Instructor for online Course P/1 seminar: http://smartURL.it/onlineactuary
Exercise for July 30, 2005

At a party, 7 people put their coats in the back room where the coats are mixed together. Each person then randomly selects one coat. Let the random variable X be the number of people who select their own coat. Find $\operatorname{Var}(X)$.

A. 16 B. 9 C. 4 D. 1 E. $\frac{1}{4}$

Solution.

We already know that $E(X)=1$. Therefore,

$$\Pr(X=0)\cdot(0-E(X))^2=\frac{1}{0!}\cdot\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}-\frac{1}{7!}\right)\cdot(0-1)^2,$$

$$\Pr(X=1)\cdot(1-E(X))^2=\frac{1}{1!}\cdot\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}\right)\cdot(1-1)^2,$$

$$\Pr(X=2)\cdot(2-E(X))^2=\frac{1}{2!}\cdot\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right)\cdot(2-1)^2,$$

$\ldots$

$$\Pr(X=3)\cdot(3-E(X))^2=\frac{1}{3!}\cdot\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right)\cdot(3-1)^2,$$

$$\Pr(X=4)\cdot(4-E(X))^2=\frac{1}{4!}\cdot\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}\right)\cdot(4-1)^2,$$

$$\Pr(X=5)\cdot(5-E(X))^2=\frac{1}{5!}\cdot\left(1-\frac{1}{1!}+\frac{1}{2!}\right)\cdot(5-1)^2,$$

$$\Pr(X=6)\cdot(6-E(X))^2=\frac{1}{6!}\cdot\left(1-\frac{1}{1!}\right)\cdot(6-1)^2,$$

$$\Pr(X=7)\cdot(7-E(X))^2=\frac{1}{7!}\cdot1\cdot(7-1)^2.$$

Therefore,

$$\operatorname{Var}(X)=\frac{103}{280}\cdot1+\frac{53}{144}\cdot0+\frac{11}{60}\cdot1+\frac{1}{16}\cdot4+\frac{1}{72}\cdot9+\frac{1}{240}\cdot16+0\cdot25+\frac{1}{5040}\cdot36=$$
$$=\frac{103}{280}+\frac{11}{60}+\frac{1}{4}+\frac{1}{8}+\frac{1}{15}+\frac{1}{140}=\frac{309+154+210+105+56+6}{840}=1.$$

Answer D.

In the general case

$$\Pr(Y=0)\cdot0^2=\frac{1}{0!}\cdot\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\ldots+(-1)^n\frac{1}{n!}\right)\cdot0^2,$$

$$\Pr(Y=1)\cdot1^2=\frac{1}{1!}\cdot\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\ldots+(-1)^{n-1}\frac{1}{(n-1)!}\right)\cdot1,$$

$$\Pr(Y=2)\cdot 2^2=\frac{1}{2!}\cdot\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\ldots+(-1)^{n-2}\frac{1}{(n-2)!}\right)\cdot 2^2,$$

$\ldots$

$$\Pr(Y=2)\cdot 2^2=\frac{1}{k!}\cdot\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\ldots+(-1)^{n-k}\frac{1}{(n-k)!}\right)\cdot k^2,$$

$\ldots$

$$\Pr(Y=n-1)\cdot(n-1)^2=0\cdot(n-1)^2,$$

$$\Pr(Y=n)\cdot n^2=\frac{1}{n!}\cdot n^2.$$

Therefore, the second moment of this random variable is

$$\frac{1}{0!}\cdot\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\ldots+(-1)^{n}\frac{1}{n!}\right)\cdot 0^2+\frac{1}{1!}\cdot\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\ldots+(-1)^{n-1}\frac{1}{(n-1)!}\right)\cdot 1^2+$$

$$\ldots+\frac{1}{k!}\cdot\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\ldots+(-1)^{n-k}\frac{1}{(n-k)!}\right)\cdot k^2+\ldots+0\cdot(n-1)^2+\frac{1}{n!}\cdot n^2=$$

$$=\sum_{k=1}^{n}\frac{k}{(k-1)!}\cdot\left(\frac{1}{0!}-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\ldots+(-1)^{n-k}\frac{1}{(n-k)!}\right).$$

But

$$\sum_{k=1}^{n}\frac{k}{(k-1)!}\cdot\left(\frac{1}{0!}-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\ldots+(-1)^{n-k}\frac{1}{(n-k)!}\right)=$$

$$=\sum_{k=1}^{n}\frac{(k-1)+1}{(k-1)!}\cdot\left(\frac{1}{0!}-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\ldots+(-1)^{n-k}\frac{1}{(n-k)!}\right)=$$

$$=\sum_{k=2}^{n}\frac{1}{(k-2)!}\cdot\left(\frac{1}{0!}-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\ldots+(-1)^{n-k}\frac{1}{(n-k)!}\right)+$$

$$+\underbrace{\sum_{k=1}^{n}\frac{1}{(k-1)!}\cdot\left(\frac{1}{0!}-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\ldots+(-1)^{n-k}\frac{1}{(n-k)!}\right)}_{=E(X)=1}=$$

$$=\underbrace{\sum_{k=2}^{n}\frac{1}{(k-2)!}\cdot\left(\frac{1}{0!}-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\ldots+(-1)^{n-(k-1)}\frac{1}{(n-(k-1))!}\right)}_{=E(X)=1\quad\text{(for the corresponding random variable defined for } n-1)}+$$

$$+\sum_{k=2}^{n}\frac{(-1)^{n-k}}{(k-2)!\cdot(n-k)!}+1=2+\sum_{k=2}^{n}\frac{1^{k-2}(-1)^{n-k}}{(k-2)!\cdot(n-k)!}=\underbrace{|j=k-2|}_{\text{Change of variables}}=$$

$$=2+\sum_{j=0}^{n-2}\frac{1^{j}(-1)^{(n-2)-j}}{j!\cdot((n-2)-j)!}=2+(1-1)^{n-2}=2.$$

Therefore,

$$\text{Var}(Y) = 2 - 1^2 = 1.$$

The result that $\text{Var}(Y) = 1$ is true for any n not just 7 as specified in this problem.