LINGUISTIC HEDGES AND FUZZY NORMALIZATION OPERATOR

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INTRODUCTION

In the evaluation of risk it is essential to assign to hazardous situations their risk scores, thus allowing our objective evaluation of uncertainty of loss imposed by a particular loss. Our objective is to study risk viewed as a fuzzy concept.

Traditionally, the calculation of a quantitative value of risk in industrial safety analysis, denoted by S, is based on an assignment of numerical values to its factors, considered to be likelihood of occurrence of the hazardous event (L), exposure (E), and possible consequences (C). The numerical product of these factors is the risk score, i.e., the value of S. As stated by Biller and Feagans [2], risk is better described as a fuzzy concept, as there does not exist a unique risk that a hazardous event will occur in a given period of time. Furthermore, Zimmer [11] observed that humans are unsuccessful and apprehensive in quantitative predictions (such as evaluations of likelihood, exposure, and consequences), but they may be significantly more efficient in qualitative forecasting.

Karwowski and Mital [5] proposed then that the likelihood L, exposure E, and consequences C are treated as linguistic variables [10]. The primary terms for the variable likelihood 'likely', 'possible' and 'unlikely', along with the possible use of hedges, propositional connectives and logical negation, are treated as modifiers of the operands in a context-dependent situation. The primary terms for C were 'high', 'low' and 'medium', and for E were 'rare', 'frequent', 'medium'. Derivation of risk scores S, treated as linguistic variables, was done through the process of approximate reasoning. As stated by Zadeh, approximate reasoning refers to the process by which an imprecise conclusion is deduced from a collection of imprecise premises, the reasoning being qualitative rather than quantitative in nature. The interpretation which we follow is Zadeh's maximin rule (see [10]).

FUZZY RISK SCORE

The main problem of interest here is the following question: given exposure E, consequences C, and likelihood L, what is the value of risk S? Karwowski and Mital [5] define first the fuzzy relation R between E and L as a product R_{ExL} in a matrix form. Taking the risk S to be $S = (C \circ R_{RxL}) \cap (E \circ R_{CxL}) \cap (L \circ R_{ExC})$ we obtain the fuzzy risk score.

We will discuss the difficulties that appear in interpretation of the results of approximate reasoning results, as defined above, and offer some solutions to these problems.

Example. If:

E = *medium* ={0/0, 1/0.2, 2/0.7, 3/1.0, 4/0.7, 5/0.2, 6/0}
C = *very high* = {0/0, 1/0, 2/0, 3/0.1, 4/0.5, 5/0.8, 6/1}
L = *unlikely* = {0/1, 1/1, 2/0.9, 3/0.8, 4/0.5, 5/0, 6/0}
and the value of risk is:
S = {0/0, 1/0, 2/0.1, 3/0.3, 4/0.5, 5/0.5, 6/0.5}

The interpretation of this result is somewhat a puzzle, as it is not given in standard interpretation of linguistic values. Schmucker in [8] derives an alternative expression of risk as a weighted average of its factors, and encounters a similar problem, handling it through the standard normalization process. We want to present another possible way of interpreting values of linguistic variables as the one for S above. Let us note, for example, that, to a certain degree, and if the scaling is appropriate, if the number 6 belongs to a fuzzy set describing *high risk* with degree of membership 0.5, we can approximate this by having 3 in the set with membership value 1.0. This is a form of switching from fuzzy to crisp environments, relaxed to apply to fuzzy sets.

Based on this, we propose the following alternative for standard normalization, which we will call the *fuzzy normalization operator*.

FUZZY NORMALIZATION OPERATOR

Let $A = \{x/f_c(x)\}$ be a fuzzy set such that for some $\alpha \in (0,1)$, $\alpha < 1$, its α-cut A_a is empty. This is precisely the situation which appears to cause difficulties in interpretation. Let then a new membership function be introduced, we will denote it by g_c, in the following manner:

$$g_c(x) = f(kx) / f(k)$$

where k is the largest value of x at which f attains its maximum. Obviously, the support of g_c is then contained in [0, M/k], where M is the upper bound of the support of f_c, but we can simply put $g_c(x) = 0$ for $x > M/k$.

This process provides a more intuitively acceptable interpretation of results of approximate reasoning in certain cases.

Consider the risk score S:
S = {0/0, 1/0, 2/0.1, 3/0.3, 4/0.5, 5/0.5, 6/0.5}.
Applying the fuzzy normalization operator we obtain
S' = {0/0,0.5/0,1/0.2,1.5/0.6,2/1,2.5/1,3/1,4/0, 5/0, 6/0}.

The noninteger values of x appearing in the result can be handled in the following manner. For any integer n, we can take $g_c(n)$ to be the arithmetic average of all values of $g_c(x)$, where n-1<x<n+1. This modification changes S' to S' = {0/0, 1/0.267, 2/0.867, 3/1, 4/.0, 5/0, 6/0}. However, this modification may produce fractional values for the membership function, so we might also consider an alternative $g_c''(n) = \vee \{ g_c(x) : n - 1 < x < n + 1 \}$ which will produce

S" = {0/0, 1/0, 2/0.6, 3/1, 4/0, 5/0, 6/0, 7/0}.

Also, one may apply the standard normalization operator (see [8] p. 13) to S', if necessary (this is not the case here). Another alternative is:

$$g_c'''(n) = \wedge \{ g_c(x) : n - 1 < x < n + 1 \}$$

with

$$S''' = \{ 0/0, 1/0, 2/1, 3/0, 4/0, 5/0, 6/0 \}.$$

This approach switches the result to the crisp environment, which is a rather surprising result. Of the three proposed approaches, we believe the simple arithmetic average to be the most appropriate.

The interesting conclusion is that risk scores S', S'' and S''' obtained here can easily be interpreted as *medium*, which is a more reasonable result than the interpretation based on standard normalization procedure, as *more-or -less high*. Clearly, *medium* exposure with *very high* consequences but low likelihood should produce *medium* risk. We may give here an example of an amateur parachute jumper who performs the jumps under the supervision of a qualified instructor. Such condition results in being qualified as substandard risk by life insurance companies, and issuing rated policies, but not in denying insurance (which would imply that insurance companies consider the situation *high risk*). Let us notice that if the domain of the membership function is continuous then fuzzy normalization procedure does not require taking the arithmetic average at the end.

In relation to the fuzzy normalization operator presented here, we would like to discuss a possibility of applying it in an alternative definition of the concentration and dilation operators, used to define the hedges *very* or *more* or *less*.

Consider the value *high* of a linguistic variable given: *high* ={0/0, 2/0.1, 3/0.3, 4/0.7, 5/0.9, 6/1.0}.

Then the concentration operator CON, as defined in [8] p.11,

$$CON(A) = \{ x/f_c^2(x) \}, \text{ where } A = \{ x/f_c(x) \}$$

produces *very high* = CON (*high*)

very high = { 0/0, 2/0.1, 3/0.09, 4/0.49, 5/0.81, 6/1.03 }, or after rounding off

very high = { 0/0, 2/0, 3/0.1, 4/0.5, 5/0.8, 6/1.0 }.

However, as Schmucker points out in [7] pp. 40-41, in his interesting discussion of the problem, the traditional operator CON may not always be an appropriate model for the hedge *very*. Hersh and Caramazza in [4] and Macvicar-Whelen in [6] exhibited experimental data demonstrating that the CON operator may need to be replaced, at least in certain cases, by a shift by a certain constant **k**, i.e., if $A = \{ x/f_c(x) \}$ then *very* $A = \{ x/f_c(x-k) \}$.

There are, however, certain difficulties with applications of such an operator. Undoubtedly, a person who is 50 years old has a positive degree of membership in the set of *old* people, one would doubt however any positive degree of membership in the set of *very old* people. Thus a shift may be appropriate. On the other hand, the shifted membership function may assign positive degree of membership to elements outside of the support of a given fuzzy set. Also, it seems inappropriate to model *very average* or similar expression with the hedge *very* defined by a shift operator.

Finally, the size of the shift is not clearly defined. This, however, may be advantageous when dealing with qualitative statements describing risk and its factors, or other fuzzy variables.

Let us propose another approach, which seems to be a compromise between the two. Notice the relationship between *high* and *very high*: the α-cuts of *very high* contain the α-cuts of *high*, and if compatibility functions of *high* and *very high* are graphed, the slope of the one corresponding to *very high* is steeper. Thus it would seem appropriate to do the following: If a value of a linguistic variable:

$V = \{x/f_c(x)\}$ is given, and 1 the maximum value of f_c (height of f_c) is at the right-hand endpoint M of its domain, let

very $\{ x/f_c(x) \} = \{ x/(g_c(2x))^2 \}$

where $g_c(x) = f_c(M-x)$, $0 \le x \le M$.

If 1, the maximum of f_c is at the left-hand endpoint of its domain, let the new compatibility function be simply $(f_c(2x))^2$. Finally, if there is a maximum of 1 at a point in the interior of the domain of f_c, apply the above operations separately to the left of that point and to the right. Figure 1 shows graphical representation of $f_c(x)$ and $g_c(x)$.

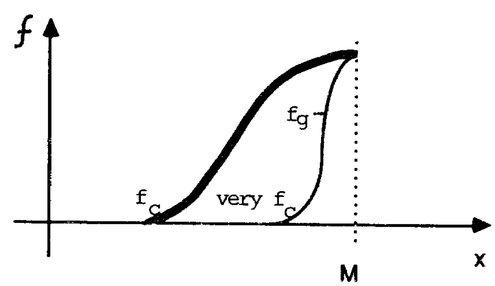

Figure 1. Representation of the operator *very*.

Also, if
high = {0/0, 1/0, 2/0.1, 3/0.3, 4/0.7, 5/0.9, 6/1.0}
then *very high* defined using the definition proposed here will be: *very high* = {0/0, 1/0, 2/0, 3/0, 3.5/0.05, 4/0.15, 4.5/0.49, 5.5/0.81, 6/1.0}
which we can average to be: *very high* = {0/0, 1/0, 2/0, 3/0.025, 4/0.23, 5/0.65, 6/0.905}, and after standard normalization and rounding off: *very high* = {0/0, 1/0, 2/0, 3/0.03, 4/0.25, 5/0.72, 6/1.0}.
Alternatively, taking minima over neighboring fractions *very high* = { 0/0, 1/0, 2/0, 3/0, 4/0.05, 5/0.49, 6/0.81 } and standard normalization gives:
very high = {0/0,1/0, 2/0, 3/0, 4/0.06, 5/0.60, 6/1.0}.
Taking maxima over neighboring fractions results in:
very high = {0/0, 1/0, 2/0, 3/0.05, 4/0.49, 5/0.81, 6/1.0}.
The last result is almost identical with the result of an application of the traditional CON operator *very high* = CON(*high*) = { 0/0, 1/0, 2/0, 3/0.1, 4/0.5, 5/0.8, 6/1.0}.

The alternative definition of the hedge *very* seems to be especially appropriate when an application of the hedge *very* is an intermediate step between fuzzy and crisp environments.

If we look at Figure 1 and decide to switch to a crisp set with characteristic function

$$X(x) = \{ \begin{smallmatrix} 1 & x=M \\ 0 & x<M \end{smallmatrix} \}$$

then

$$X(x) = \lim_{n \to \infty} (f(n(M-x)))^n .$$

FUZZY INVESTMENT ANALYSIS

One of the document theories in the asset allocation is the Modern Portfolio Theory (MPT), created by H. Markowitz [7]. MPT identifies risk with the standard deviation of returns on an investment. Then it seeks efficient frontiers consisting of portfolios maximizing expected return given a degree of risk. Markowitz was in fact the first theoretician to propose the general approach of maximizing return while minimizing risk.

On the other hand, the specific hazards that investors are trying to avoid are not exactly due to high standard deviation. One can, in fact, give a simple example of a game in which standard deviation of return is high while no risk exists: Let A flip a coin and pay B $1.00 if tails is observed, $2.00 if heads is observed, and let B pay $1.00 for the right to play this game; then the expected return is 50% and standard deviation is 50% (for a fair coin).

It is more reasonable, then to look at hazards facing an investor. Assume that we are dealing with a stock market investor.

When dealing with risk in industrial safety we were able to specify certain major risk factors and then calculate risk scores based on assessments of linguistic values of risk factors. Let us notice that there seems to be a trend towards analysis of risk in terms of its factors (factors C, L, E in industrial risk analysis and risks C-1, C-2, C-3 utilized by the Society of Actuaries, see [1], where C-1 represents what can be described as interest rate risk, C-2 - catastrophe risk, and C-3 - investment and management risk).

The main hazard facing an investor is a significant decline in the purchasing power value of his assets. This can happen as a result of one of the following three events:

(i) inability to sell the asset at a desired price;
(ii) decline in the market value of the asset because of relative attractiveness of alternative investments;
(iii) loss of purchasing power through inflation.

The above three may be listed as three types of risks, which we will treat as linguistic variables

(i) L - liquidity risk;
(ii) E - equity rate risk;
(iii) I - interest rate risk.

Our major interest is in creating investment analysis expert systems. It is clear that the rules of both the fundamental and technical investment analysis are of fuzzy nature. That fuzzy nature has become in fact an object of criticism. However, there seems to exist empirical evidence of applicability of such rules, and so the study of them is desired.

There can be several technical and fundamental rules concerning the liquidity risk listed. We will choose only one, the comparison of the moving average of an appropriately chosen stock index (as Dow Jones Industrials, or Standard and Poor 500) to its current level. The technical rule we refer to states that if a certain (e.g., 40 weeks) moving average falls below the current price, the prices are likely to go up (i.e., the risk is low), if it moves above the current price, the prices are likely to go down (i.e., the risk is high). Let us then say that:

L = *high* if the moving average exceeds the current price by 5% of the current price, or less,
L = *low* if the current price exceeds the moving average by 5% of the current price, or less,
L = *medium* otherwise, and:
E = *high* if the discount and prime rates have been increased at least three times in a row,
E = *low* if the discount and prime rates have been lowered at least three times in a row,
E = *medium* otherwise, and:
I = *low* if the current price of gold exceeds the moving average by 5% of the current price or less,
I = *high* if the moving average exceeds the current price of gold by 5% of the current price or less,
I = *medium*, otherwise.
This can be, of course, treated only as an initial proposal, subject to further study. We can, however, within that simple framework, derive certain risk scores S given L, E, and I and discuss the value of our model.

Consider for example with:
L = *high* = {0/0, 1/0, 2/0.1, 3/0.3, 4/0.7, 5/0.9, 6/1.0},

E = *medium* ={0/0, 1/0.2, 2/0.7, 3/1.0, 4/0.7, 5/0.2,6/0},
I = *high* = {0/0, 1/0, 2/0.1, 3/0.3, 4/0.7, 5/0.9, 6/1.0}.
Then with S defined as:
$S = L \circ R_{ExI} \cap E \circ R_{LxI} \cap I \circ R_{LxE}$, where R_{ExI} is represented by the matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0 & 0 & 0.1 & 0.3 & 0.7 & 0.7 & 0.7 \\ 0 & 0 & 0.1 & 0.3 & 0.7 & 0.9 & 1.0 \\ 0 & 0 & 0.1 & 0.3 & 0.7 & 0.7 & 0.7 \\ 0 & 0 & 0.1 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In a similar way, we define fuzzy relations R_{LxI} and R_{LxE}, which are then used to find S, where:
$L \circ R_{ExI}$ = { 0/0, 1/0, 2/0.1, 3/0.3, 4/0.7, 5/0.7, 6/0.7 }
$E \circ R_{LxI}$ = { 0/0, 1/0, 2/0.1, 3/0.3, 4/0.7, 5/0.7, 6/0.7 }
$I \circ R_{LxE}$ = { 0/0, 1/02, 2/0.7, 3/1.0, 4/0.7, 5/0.2, 6/0 },
and S = { 0/0, 1/0, 2/0.1, 3/0.3, 4/0.7, 5/0.2, 6/0 }.

Fuzzy normalization applied to S gives:
S' = (0/0, 2.5/0, 3/0.14, 3.5/0.42, 4/1, 4.5/0.28, 5/0)
and averaging over neighboring fractions gives:
S'' = {0/0, 1/0, 2/0, 3/0.19, 4/0.57, 5/0.14, 6/0},
and finally standard normalization gives:
S''' = { 0/0, 1/0, 2/0, 3/0.33, 4/1, 5/0.24, 6/0 }.

This risk score can be interpreted as *somewhat high but not very high*. Now consider the implication of an increase in discount rate, or prime rate on this situation, where:
E = *high* = { 0/0, 1/0, 2/0.1, 3/0.3, 4/0.7, 5/0.9, 6/1.0 }
and $R_{ExI} = R_{LxI} = R_{LxE}$ are represented by the following fuzzy relation:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0 & 0 & 0.1 & 0.3 & 0.3 & 0.3 & 0.3 \\ 0 & 0 & 0.1 & 0.3 & 0.7 & 0.7 & 0.7 \\ 0 & 0 & 0.1 & 0.3 & 0.7 & 0.9 & 0.9 \\ 0 & 0 & 0.1 & 1.3 & 0.7 & 0.9 & 1.0 \end{bmatrix}$$

Notice that in the above case, the following is true:

$L \circ R_{ExI} = E \circ R_{LxI} = I \circ R_{LxE}$ = { 0/0, 1/0, 2/0.1, 3/0.3, 4/0.7, 5/0.9, 6/1.0 } = *high*.

We believe this to be quite an interesting model of the development in the financial markets in September 1987 and October 1987 when *somewhat high but not very high* could be an appropriate description of the risk in the summer of 1987, but the discount rate hike changed it to *high*.

CONCLUSIONS

Numerous rules of fundamental and technical investment analysis are in fact rather fuzzy in nature. A consistent application of approximate reasoning in that area can bring a successful extension of the theory of expert systems to the field of investment analysis.

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