The space of Henstock integrable functions II

Piotr Mikusiński, and Krzysztof Ostaszewski

Abstract

The space of Henstock integrable functions on the unit cube in the m-dimensional Euclidean space is normed, barrelled, and not complete. We describe its completion in the space of Schwartz distributions.

We also show how the distribution functions for finite signed Borel measures are multipliers for the Henstock integrable functions, and how they generate continuous linear functionals on the space of Henstock integrable functions. Finally, we discuss various integration by parts formulas for the two-dimensional Henstock integral.

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1.1. Definition. Let $I_0 \subset I\!R^m$ be the unit cube in the m-dimensional Euclidean space. A function $f : I_0 \to I\!R$ will be termed ***Henstock integrable***, with

$$\int\int \cdots \int_{I_0} f(x_1, x_2, \ldots, x_m) dx_1 dx_2 \ldots dx_m \tag{1}$$

written for the value of the integral, if for every $\varepsilon > 0$ there exists a positive function $\delta : I_0 \to I\!R$ (usually called a ***gauge***) such that whenever

$$\pi = \{((x_1^i, x_2^i, \ldots, x_m^i), I_i) : i = 1, 2, \ldots, n\} \tag{2}$$

is a partition of I_0, consisting of pairs of points in I_0 and nonoverlapping subintervals of I_0 whose union is the whole I_0, and such that for every $i = 1, 2, \ldots, n$, $(x_1^i, x_2^i, \ldots, x_m^i) \in I_i$ and I_i is contained in the ball centered at $(x_1^i, x_2^i, \ldots, x_m^i)$ of radius $\delta(x_1^i, x_2^i, \ldots, x_m^i)$, we have

$$|\sum_{i=1}^{n} f(x_1^i, x_2^i, \ldots, x_m^i)\lambda(I_i) - \int\int\ldots\int_{I_0} f(x_1, x_2, \ldots, x_m)dx_1dx_2\ldots dx_m| < \varepsilon; \tag{3}$$

here λ stands for the m-dimensional volume of an interval $I \subset \mathbb{R}^m$. Quite often we will simply write

$$\int_{I_0} f d\lambda \tag{4}$$

for the Henstock integral of f over I_0.

A partition π as in (2), satisfying conditions listed between (2) and (3) will be called ***δ-fine***.

We will denote by $\tilde{f}$ the indefinite Henstock integral of a function f, i.e.,

$$\tilde{f}(x_1, x_2, \ldots, x_m) = \int_0^{x_1}\int_0^{x_2}\ldots\int_0^{x_m} f(t_1, t_2, \ldots, t_m)dt_1dt_2\ldots dt_m =$$

$$\int\int\ldots\int_{I_0} f(t_1, t_2, \ldots, t_m)\chi_{[0,x_1]\times[0,x_2]\times[0,x_m]}dt_1dt_2\ldots dt_m, \tag{5}$$

where χ_E denotes the characteristic function of set $E \subset \mathbb{R}$.

If H is an interval function and we replace λ by H in (3) then we get the concept of the Henstock integral of f with respect to H, written as

$$\int\int\ldots\int_{I_0} f dH \text{ or just } \int_{I_0} f dH. \tag{6}$$

If $g:\ I_0 \to \mathbb{R}$ then it generates an interval function H as follows. Let

$$I = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_m, b_m], \tag{7}$$

define

$$H(I) = \sum_{J\subset\{1,2,3,\ldots,m\}} (-1)^{\mathrm{card}J} g(c_1, c_2, \ldots, c_m), \tag{8}$$

where

$$\{c_1, c_2, \ldots, c_m\} = \{a_{j_1}, a_{j_2}, \ldots a_{j_k}, b_{i_1}, b_{i_2}, \ldots, b_{i_l}\}, \tag{9}$$

and

$$\{j_1, j_2, \ldots, j_k\} = J,\ \{i_1, i_2, \ldots, i_l\} = \{1, 2, \ldots, m\} \setminus J, \tag{10}$$

and card$J$ is the cardinality of J. The integral

$$\int\int\ldots\int_{I_0} f dH \tag{11}$$

is called the Henstock integral of f with respect to g and written as

$$\int\int\cdots\int_{I_0} fdg \text{ or simply } \int_{I_0} fdg. \tag{12}$$

Please consult [2], [4], [5], and [7] for these definitions.

1.2. The class of Henstock integrable functions on I_0 will be denoted by $\mathcal{H}$. It is a linear topological space. In [8] and [9] it is shown that the space equipped with the Alexiewicz norm is barrelled, but it is not a Banach space. [6] and [8] discuss the dual of the space. The work in [8] is done in the two-dimensional case, but easily extends to the multidimensional one. [6] considers the dual of $\mathcal{H}$ for functions of one variable.

Our intention is to describe the completion of the space and to further discuss its dual.

Let us note that every Henstock integrable function $f : I_0 \to \mathbb{R}$ is a Schwartz distribution (see [4], section 2.12).

1.3. Definition. Denote by $\mathcal{F}$ the space of all ***distributions of order m*** with support in I_0, i.e., $f \in \mathcal{F}$ if there exists a continuous function $F : \mathbb{R}^m \to \mathbb{R}$ such that

$$F(x_1, x_2, \ldots, x_m) = 0 \text{ if } \min\{x_1, x_2, \ldots, x_m\} \leq 0, \tag{13}$$

$$F(x_1, x_2, \ldots, x_i, \ldots, x_m) = F(x_1, x_2, \ldots, 1, \ldots, x_m) \text{ if } x_i \geq 1 \tag{14}$$

for $i = 1, 2 \ldots, m$,

$$f = \frac{\partial^m F}{\partial x_1 \partial x_2 \ldots \partial x_m}, \tag{15}$$

where the derivatives are understood in the distributional sense.

For $f \in \mathcal{F}$ as in (13), (14), and (15) define

$$\int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_m} f(t_1, t_2, \ldots, t_m)dt_1 dt_2 \ldots dt_m = F(x_1, x_2, \ldots, x_m) \tag{16}$$

for $(x_1, x_2, \ldots, x_m) \in I_0$. Note that, for every $f \in \mathcal{F}$ there exists exactly one function F satisfying (13), (14), and (15). Thus the integral (16) is uniquely defined. Moreover

$$||f|| = \sup_{(x_1, x_2, \ldots, x_m) \in I_0} |F(x_1, x_2, \ldots, x_m)| \tag{17}$$

is a norm on $\mathcal{F}$. We will call it the ***Alexiewicz norm***, as it is the same as the Alexiewicz norm introduced in [8] on the space of Henstock integrable functions.

1.4. Proposition. *$\mathcal{F}$ is complete.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{F}$. Let

$$F_n(x_1, x_2, \ldots, x_m) = \int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_m} f_n(t_1, t_2, \ldots, t_m)dt_1dt_2\ldots dt_m. \quad (18)$$

Then $\{F_n\}$ is a Cauchy sequence (with respect to the sup norm) of continuous functions satisfying (13), (14), and (15), therefore it is convergent to a continuous function F satisfying the same conditions. Define

$$f = \frac{\partial^m F}{\partial x_1 \partial x_2 \ldots \partial x_m}. \quad (19)$$

Then $f \in \mathcal{F}$ and $\lim_{n\to\infty} ||f_n - f|| = 0$.

1.5. Observation. $\mathcal{H} \subset \mathcal{F}$.

Proof. This immediately follows from the following statement proved in [4], sections 2.3, and 2.12: if $f \in \mathcal{H}$ then $\tilde{f}$ is continuous and if $\phi : I_0 \to I\!R$ is m times continuously differentiable and

$$\psi(x_1, x_2, \ldots, x_m) = \frac{\partial^m \phi(x_1, x_2, \ldots, x_m)}{\partial x_1 \partial x_2 \ldots \partial x_m} \quad (20)$$

then

$$\int\int\cdots\int_{I_0} f(x_1, x_2, \ldots, x_m)\phi(x_1, x_2, \ldots, x_m)dx_1dx_2\ldots dx_m =$$

$$(-1)^m \int\int\cdots\int_{I_0} \tilde{f}(x_1, x_2, \ldots, x_m)\psi(x_1, x_2, \ldots, x_m)dx_1dx_2\ldots dx_m. \quad (21)$$

1.6.Theorem. *$\mathcal{F}$ is the completion of $\mathcal{H}$.*

Proof. Denote by $\mathcal{H}^*$ the completion of $\mathcal{H}$ with respect to the Alexiewicz norm. Then $\mathcal{H}^* \subset \mathcal{F}$ and both spaces are complete with respect to the same norm. Therefore, by the open mapping theorem, they are equal.

1.7. Remark. In the one-dimensional case it is known that every Henstock integrable function is almost everywhere a derivative of its indefinite integral. This implies that in that case, $\mathcal{H}$ is of the first category in $\mathcal{F}$. An easy example of an element of $\mathcal{F}$ which is not in $\mathcal{H}$ is in that case a distributional derivative of a nowhere differentiable continuous function.

2.1. We will turn now to our discussion of the dual of the space $\mathcal{H}$. We have the following, as presented in [6] and [8]:

In the one-dimensional case T is a continuous linear functional on $\mathcal{H}$ if and only if either of the following holds (all integrals used below are Henstock integrals):

(a) There exists a finite signed Borel measure μ_T on $(0, 1]$ *such that*

$$T(f) = \int_0^1 \tilde{f}(t)d\mu_T(t), \quad (22)$$

where, as usual

$$\tilde{f}(x) = \int_0^x f(t)dt. \tag{23}$$

(b) There exists a function $g_T : [0,\ 1] \to I\!R$ *of essentially bounded variation such that*

$$T(f) = \int_0^1 f(t)g_T(t)dt. \tag{24}$$

Being of essentially bounded variation is equivalent to having a signed finite Borel measure as a distributional derivative. If μ_g stands for that distributional derivative then integration by parts yields

$$\int_0^1 \tilde{f}(t)d\mu_T(t) = \int_0^1 \tilde{f}(t)d\mu_g(t) + \tilde{f}(1)\mu_g((0,\ 1]). \tag{25}$$

Notice that the expression $\tilde{f}(1)\mu_g((0,\ 1])$ is itself a continuous linear functional of f.

As observed in [8] the description (a) easily extends to the multidimensional case. However, (b) uses the class of multipliers for the Henstock integrable functions (i.e., functions which multiplied by a Henstock integrable function produce a Henstock integrable function), which is not known in the multidimensional case.

For simplicity, let us restrict ourselves to the two-dimensional case, with $I_0 = [0,\ 1] \times [0,\ 1]$. This does not affect generality of the results.

2.2. Definition A function $g : I_0 \to I\!R$ is of ***strongly bounded variation*** (see[4]) if for every $x \in [0,\ 1]$, $g(x,\ \cdot)$ is of bounded variation, for every $y \in [0,\ 1]$, $g(\cdot,\ y)$ is of bounded variation, and

$$\sup \sum_{i=1}^{n} |g(a_i,\ c_i) - g(a_i,\ d_i) - g(b_i,\ c_i) + g(b_i,\ d_i)| < +\infty, \tag{26}$$

where the least upper bound is taken over all partitions of I_0 into a finite collection of nonoverlapping nondegenerate closed intervals $[a_i,\ b_i] \times [c_i,\ d_i]$, $i = 1,\ 2,\ 3,\ \dots, n$.

Let us note that [4] contains the definition of a function of strongly bounded variation in the general multidimensional case.

2.3. Theorem. *Every function of strongly bounded variation is a multiplier for Henstock integrable functions.*

Proof. See[4].

2.4. It is not known whether the above is a complete characterization of multipliers. Our intention is to point out a specific subclass of the class of functions of strongly bounded variation.

2.5. Definition. Let $\mathcal{D}$ stand for the class of two-dimensional distribution functions of finite signed Borel measures on $(0\ ,1] \times (0,\ 1]$. For example, if μ is a positive measure then $g_\mu \in \mathcal{D}$ corresponding to it is

$$g_\mu(x, y) = \mu((0, x] \times (0, y]). \tag{27}$$

The value of $g_\mu(x, y)$ for $x = 0$ or $y = 0$ is inessential to us. We will assume it to be zero.

In general, for a signed finite Borel measure μ on $(0, 1] \times (0, 1]$ we will denote its distribution function by g_μ.

Also, we will denote by $\mathcal{M}$ the class of finite signed Borel measures on $(0, 1] \times (0, 1]$. $\mathcal{M}^+$ will denote the class of positive measures in $\mathcal{M}$.

2.6. Proposition. *The elements of $\mathcal{D}$ are of strongly bounded variation.*

Proof. It suffices to show that for a $\mu \in \mathcal{M}^+$, g_μ is of strongly bounded variation. One can easily see that both $g_\mu(x, \cdot)$ and $g_\mu(\cdot, y)$ are monotone for every $x \in [0, 1]$ and every $y \in [0, 1]$, so that they are of bounded variation. Let $\{I_i : i = 1, 2, \ldots, n\}$ be a finite class of nonoverlapping nondegenerate subintervals of I_0, and $I_i = [a_i, b_i] \times [c_i, d_i]$ for $i = 1, 2, \ldots, n$. Consider the sum

$$\sum_{i=1}^{n} |g_\mu(b_i, d_i) - g_\mu(a_i, c_i) - g_\mu(b_i, d_i) + g_\mu(a_i, b_i)| =$$

$$\sum_{i=1}^{n} |\mu((a_i, b_i] \times (c_i, d_i])| \leq ||\mu||. \tag{28}$$

This implies that g_μ is of strongly bounded variation.

2.7. Corollary. *A distribution function of a finite signed Borel measure is a multiplier for Henstock integrable functions.*

2.8. Corollary. *If $g : I_0 \to \mathbb{R}$ is equivalent to a distribution function of a finite signed Borel measure then g is a multiplier for Henstock integrable functions.*

2.9. Definition. Let $\mathcal{C}_0$ denote the class of all continuous $F : I_0 \to \mathbb{R}$ such that F is continuous and $F(x, y) = 0$ whenever $x = 0$ or $y = 0$.

2.10. Observation. *If $F \in \mathcal{C}_0$) and $\mu \in \mathcal{M}$ then the Lebesgue-Stieltjes integral*

$$\mathcal{L}\int\int_{I_0} F d\mu \tag{29}$$

exists and is well-defined.

2.11. Proposition. *The Riemann-Stieltjes integral of $F \in \mathcal{C}_0$ with respect to a $\mu \in \mathcal{M}$, denoted by*

$$\mathcal{R}\int\int_{I_0} F d\mu, \tag{30}$$

is naturally defined as the limit of the Riemann sums

$$\sum_{i=1}^{n} F(x_i, y_i)\mu((a_i, b_i] \times (c_i, d_i]), \tag{31}$$

where the elements of $\{(a_i, b_i] \times (c_i, d_i] : i = 1, 2, \ldots, n\}$ ***are disjoint, their union is*** $(0, 1] \times (0, 1]$, ***and*** $(x_i, y_i) \in (a_i, b_i] \times (c_i, d_i]$ ***for*** $i = 1, 2, \ldots, n$. ***The limit is taken with respect to the norm of the partition, which is the diameter of the largest of the intervals in the partition, tending to zero.***

Existence of

$$(\mathcal{R}) \int \int_{I_0} F d\mu \tag{32}$$

implies existence of the Henstock integral of F ***with respect to*** g_μ ***and their equality.***

Proof. It suffices to consider $\mu \in M^+$. Let $\varepsilon > 0$ be arbitrary. Choose a $\delta > 0$ such that whenever the norm of the partition

$$\{(a_i, b_i] \times (c_i, d_i] : i = 1, 2, \ldots,\} \tag{33}$$

is less than δ, and (x_i, y_i) belongs to $(a_i, b_i] \times (c_i, d_i]$ for every $i = 1, 2, \ldots, m$, we have

$$|\sum_{i=1}^{n} F(x_i, y_i)\mu((a_i, b_i] \times (c_i, d_i]) - (\mathcal{R}) \int \int_{I_0} F d\mu| < \varepsilon \tag{34}$$

Now let $p : I_0 \to I\!R$ be a gauge function defined as follows:

$$\begin{aligned} p(x, y) &= \delta/2 \text{ if } (x, y) \in (0, 1] \times (0, 1], \\ \text{and } &= 2 \text{ if } (x, y) \notin (0, 1] \times (0, 1]. \end{aligned} \tag{35}$$

If

$$\pi = \{((x_i, y_i), (a_i, b_i] \times (c_i, d_i]) : i = 1, 2, \ldots, n\} \tag{36}$$

is a p-fine partition, then its norm is less than δ and

$$\sum_{i=1}^{n} F(x_i, y_i)(g_\mu(a_i, c_i) - g_\mu(a_i, d_i) - g_\mu(b_i, c_i) + g\mu(b_i, d_i)) =$$

$$\sum_{i=1}^{n} F(x_i, y_i)\mu((a_i, b_i] \times (c_i, d_i]). \tag{37}$$

Consequently, the Henstock integral of F with respect to g_μ exists and equals

$$(\mathcal{R}) \int \int_{I_0} F d\mu \tag{38}$$

2.12. Proposition. ***If*** $F \in \mathcal{C}_0$ ***and*** $\mu \in \mathcal{M}$ ***then***

$$(\mathcal{R}) \int \int_{I_0} F d\mu \tag{39}$$

exists.

Proof. Let $\varepsilon > 0$ be arbitrary. Choose a number η such that

$$\eta < \frac{\varepsilon}{2\mu((0,\ 1] \times (0,\ 1])}. \tag{40}$$

There exists a $\delta > 0$ such that if $(x_1,\ y_1),(x_2,\ y_2) \in I_0$ and the distance from $(x_1,\ y_1)$ to $(x_2,\ y_2)$ is less than δ then $|F(x_1,\ y_1) - F(x_2,\ y_2)| < \eta$. Let

$$\{((x_i,\ y_i),\ I_i) :\ i = 1,\ 2,\ \ldots,\ n\},\ \ \{((s_j,\ t_j),\ J_j) :\ j = 1,\ 2,\ \ldots,\ r\} \tag{41}$$

be two partitions of I_0, both with norm less than δ. Consider the intervals

$$P_{i,j} = I_i \cap J_j, i = 1,\ 2,\ \ldots,\ n,\ j = 1,\ 2,\ \ldots,\ r. \tag{42}$$

In each nonempty $P_{i,j}$ choose a point $(u_{i,j}, v_{i,j})$. Then the distances between $(u_{i,j}, v_{i,j})$ and $(x_i,\ y_i)$, and between $(u_{i,j},\ v_{i,j})$ and $(s_j,\ t_j)$ are both less than δ. Thus

$$\begin{gathered}
|\sum_{i,j=1}^{n,r} F(u_{i,j},\ v_{i,j})\mu(P_{i,j}) - \sum_{i=1}^{n} F(x_i,\ y_i)\mu(I_i)| = \\
|\sum_{i=1}^{n}(\sum_{j=1}^{r} F(u_{i,j},\ v_{i,j})\mu(P_{i,j}) - F(x_i,\ y_i)\mu(P_{i,j}))| \leq \\
\sum_{i=1}^{n}\sum_{j=1}^{r} |F(u_{i,j},\ v_{i,j}) - F(x_i, y_i)|\mu(P_{i,j}) \leq \eta\mu(I_0) < \frac{1}{2}\varepsilon.
\end{gathered} \tag{43}$$

Similarly

$$|\sum_{i,j=1}^{n,r} F(u_{i,j},\ v_{i,j})\mu(P_{i,j}) - \sum_{j=1}^{r} F(s_j,\ t_j)\mu(J_j)| < \frac{1}{2}\varepsilon. \tag{44}$$

Therefore

$$|\sum_{i=1}^{n} F(x_i,\ y_i)\mu(I_i) - \sum_{j=1}^{r} F(s_j,\ t_j)\mu(J_j)| > \varepsilon, \tag{45}$$

which implies that

$$(\mathcal{R}) \int\int_{I_0} F d\mu \tag{46}$$

exists.

2.13. Proposition. *If $F \in \mathcal{C}_0$ and $\mu \in \mathcal{M}$ then*

$$(\mathcal{L}) \int\int_{I_0} F d\mu = (\mathcal{R}) \int\int_{I_0} F d\mu. \tag{47}$$

Proof. Let $\{\pi_k\}_{k\in\mathbf{N}}$ be a sequence of partitions of I_0,

$$\pi_k = \{((x_i^k,\, y_i^k),\, I_i^k):\ i = 1,\, 2,\, \ldots,\, n_k\} \tag{48}$$

such that the norm of the partition π_k goes to zero as $k \to \infty$. For each $k \in \mathbb{N}$ define

$$U_k(x,\, y) = \sup_{(s,t)\in I_i^k} F(s,\, t) \text{ for } (x,\, y) \in I_i^k,\ i = 1,\, 2,\, \ldots,\, n_k,$$

$$L_k(x,\, y) = \inf_{(s,t)\in I_i^k} F(s,\, t) \text{ for } (x,\, y) \in I_i^k,\ i = 1,\, 2,\, \ldots,\, n_k. \tag{49}$$

Then for every $K \in \mathbb{N}$

$$(\mathcal{L}) \int\int_{I_0} L_k d\mu \le \sum_{i=1}^{n_k} F(x_i^k,\, y_i^k)\mu(I_i^k) \le (\mathcal{L}) \int\int_{I_0} U_k d\mu. \tag{50}$$

As $k \to \infty$,

$$\sum_{i=1}^{n_k} F(x_i^k,\, y_i^k)\mu(I_i^k) \to (\mathcal{R}) \int\int_{I_0} F d\mu. \tag{51}$$

On the other hand, as $k \to \infty$

$$U_k \to F,\ L_k \to F \tag{52}$$

uniformly. This implies that

$$\lim_{k\to\infty} (\mathcal{L}) \int\int_{I_0} L_k d\mu = \lim_{k\to\infty} (\mathcal{L}) \int\int_{I_0} U_k d\mu = \int\int_{I_0} F d\mu. \tag{53}$$

(47) follows now easily from (50), (52), and (53).

2.4. Proposition. *Let $f \in \mathcal{H}$ and $\mu \in \mathcal{M}$. Then*

$$\int\int_{I_0} f(x,\, y) g_\mu(x,\, y) dx dy =$$

$$\tilde{f}(1,\, 1) g_\mu(1,\, 1) - \int_0^1 \tilde{f}(t,\, 1) d g_\mu(t,\, 1) - \int_0^1 \tilde{f}(1,\, t) d g_\mu(1,\, t) + \int\int_{I_0} \tilde{f} d\mu. \tag{54}$$

Proof. Since g_μ is of strongly bounded variation

$$\int\int_{I_0} f(x,\, y) g_\mu(x,\, y) dx dy \tag{55}$$

exists. Also, $\tilde{f} \in \mathcal{C}_0$ so that

$$\int\int_{I_0} \tilde{f} d g_\mu = (\mathcal{R}) \int\int_{I_0} \tilde{f} d\mu \tag{56}$$

exists. The formula (54) follows now from the following integration by parts formula proved by Kurzweil in [4]:

$$\int\int_{I_0} f(x,\, y)g_\mu(x,\, y)dxdy =$$

$$\int\int_{I_0} \tilde{f}(x,\, y)dg_\mu(x,\, y) - \int_0^1 \tilde{f}(t,\, 1)dg_\mu(t,\, 1)$$

$$+\int_0^1 \tilde{f}(t,\, 0)dg_\mu(t,\, 0) - \int_0^1 \tilde{f}(1,\, t)dg_\mu(1,\, t) + \int_0^1 \tilde{f}(0,\, t)dg_\mu(0,\, t) \qquad (57)$$

$$+\tilde{f}(1,\, 1)g_\mu(1,\, 1) - \tilde{f}(1,\, 0)g_\mu(1,\, 0) - \tilde{f}(0,\, 1)g_\mu(0,\, 1) - \tilde{f}(0,\, 0)g_\mu(0,\, 0).$$

Obviously, (57) combined with 2.11 yields 2.14.

2.15. Remark. For $\mu \in \mathcal{M}$ the expression

$$\int\int_{I_0} f(x,\, y)g_\mu(x,\, y)dxdy \qquad (58)$$

is a continuous linear functional on $\mathcal{H}$. We do not know, however, if (58) gives the general form of a continuous linear functional on $\mathcal{H}$. As we stated in 2.1, [8] shows that the general form of a continuous linear functional on $\mathcal{H}$ is

$$\int\int_{I_0} \tilde{f}d\mu \qquad (59)$$

where $\mu \in \mathcal{M}$. Proposition 2.14 suggests the hypothesis that (58) is in fact another general form of a continuous linear functional on $\mathcal{H}$. We were not able to either prove or disprove it.

Also the following two problems are very natural.

2.16. Problem. Given a function $g : I_0 \to \mathbb{R}$ of strongly bounded variation, is there a $\mu \in \mathcal{M}$ such that g is equivalent to g_μ?

2.17. Problem. Given a multiplier g for Henstock integrable functions, is there a $\mu \in \mathcal{M}$ such that g is equivalent to g_μ?

3.1. The Henstock integral may be defined in an abstract setting, as presented in [3] and [7] (chapter 1). The problem of characterizing the multipliers for Henstock integrable functions remains unanswered then. However, [1] contains an interesting theorem on that subject. We will show that the theorem generalizes to spaces equipped with derivation bases, in which one can define the abstract Henstock integral.

3.2. Definition. Let X be a nonempty set, and Ψ be a nonvoid class of its subsets. A nonempty class Δ contained in the powerset of $X \times \Psi$ will be termed a ***derivation base*** on X. One can take X to be $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^m$, or a locally compact Hausdorff space (these are the only settings considered until now). We will follow the notation in [7].

A ***partition*** π is a finite class consisting of elements of $X \times \Psi$ such that $D = \{I \in \Psi : (x, I) \in \pi\}$ has exactly as many elements as π and its elements are nonoverlapping (in the sense specified for X, the definition for $\mathbb{R}^m$ is the obvious one). If the union of all elements of D equals $I_0 \in \Psi$ then we say that π is a ***partition of*** I_0.

A base Δ has the ***partitioning property*** if for every $I \in \Psi$ and every $\alpha \in \Delta$ there exists a partition $\pi \subset \alpha$ of I.

If $I_0 \in \Psi$, and $F : X \times \Psi \to \mathbb{R}$ then the ***Henstock integral*** of F with respect to Δ over I_0 is a number $(\Delta)\int_{I_0} F$ such that for every ε there exists an $\alpha \in \Delta$ such that for every partition $\pi \subset \alpha$ of I_0

$$|\sum_{(x,I)\in\pi} F(x, I) - (\Delta)\int_{I_0} f| \leq \varepsilon. \tag{60}$$

Usually, we consider functions $F : X \times \Psi \to \mathbb{R}$ of the form $F(x, I) = f(x)\lambda(I)$ where $f;\ I_0 \to \mathbb{R}$ and $\lambda : \psi \to \mathbb{R}$ is additive. In this case we will write $(\Delta)\int_{I_0} f d\lambda$ for the Henstock integral of F.

Two functions $F_1 : X \times \Psi \to \mathbb{R}$, $F_2 : X \times \Psi \to \mathbb{R}$ are ***variationally equivalent*** on $I_0 \in \Psi$ if for every $\varepsilon > 0$ there exists an $\alpha \in \Delta$ and a superadditive nonnegative $\Omega : \Psi \to \mathbb{R}$ such that $\Omega(I_0) \leq \varepsilon$, and for every $(x, I) \in \alpha$ with $x \in I_0$

$$|F_1(x, I) - F_2(x, I)| \leq \Omega(I). \tag{61}$$

It is well known (see, for example, chapter 1 of [7]) that F is Henstock integrable if and only if there exists an additive $H : \Psi \to \mathbb{R}$ variationally equivalent to F. In fact, the Henstock integral

$$H(I) = (\Delta)\int_I F \tag{62}$$

is the additive function equivalent to the integrand.

For $\alpha \in \Delta$ and $E \subset X$ we define

$$\alpha[E] = \{(x, I) \in \alpha : x \in E\},\ \alpha(E) = \{(x, I) \in \alpha : I \subset E\}. \tag{63}$$

Also

$$\Delta[E] = \{\alpha[E] : \alpha \in \Delta\},\ \Delta(E) = \{\alpha(E) : \alpha \in \Delta\}. \tag{64}$$

These are called ***sections*** of the elements α of the base Δ, and of the base itself.

Δ has a ***σ-local character*** if for every sequence $\{X_n\}_{n\in\mathbb{N}}$ of disjoint subsets of X, and for every sequence $\{\beta_n\}_{n\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$, $\beta_n \in \Delta[X_n]$, there exists an $\alpha \in \Delta$ such that for every $n \in \mathbb{N}, \alpha[X_n] \subset \beta_n$.

3.3. Theorem. *Let Δ be a derivation base with the partitioning property which is also of σ-local character. Let $F: X \times \Psi \to \mathbb{R}$, $F(x, I) = f(x)\lambda(I)$, where $f: X \to \mathbb{R}$, and $\lambda: \Psi \to \mathbb{R}$ is additive, be Henstock integrable with respect to Δ on $I_0 \in \Psi$. Then for a $G: X \times \Psi \to \mathbb{R}$ of the form $G(x, I) = f(x)g(x)\lambda(I)$, where $g: X \to \mathbb{R}$, G is Henstock integrable on I_0 if and only if $K(x, I) = g(x)H(I)$, where*

$$H(I) = (\Delta)\int_I f d\lambda, \tag{65}$$

is Henstock integrable with respect to Δ on I_0.

Proof. Let $\varepsilon > 0$ be arbitrary. Since F is Henstock integrable with H being its indefinite Henstock integral, F and H are variationally equivalent. For every $n \in \mathbb{N}$ there exists an α_n such that for every $\pi \subset \alpha_n$, a partition of I_0,

$$\sum_{(x,I)\in\pi} |f(x)\lambda(I) - \int_I f d\lambda| < \frac{\varepsilon}{n2^n}. \tag{66}$$

Let

$$E_n = \{x \in I_0 : (n-1) < |g(x)| \leq n\}, \tag{67}$$

for $n \in \mathbb{N}$. Then

$$I_0 = \bigcup_{n\in\mathbb{N}} E_n. \tag{68}$$

Since Δ has a σ-local character, there exists an $\alpha \in \Delta$ such that for every $n \in \mathbb{N}$

$$\alpha[E_n] \subset \alpha_n. \tag{69}$$

If $\pi \subset \alpha$ is a partition of I_0 then

$$|\sum_{(x,I)\in\pi} f(x)g(x)\lambda(I) - \sum_{(x,I)\in\pi} g(x)(\Delta)\int_I f d\lambda| \leq$$

$$|\sum_{(x,I)\in\pi} (f(x)g(x)\lambda(I) - g(x)(\Delta)\int_I f d\lambda)| \leq$$

$$\sum_{(x,I)\in\pi} |g(x)||f(x)\lambda(I) - \int_I f d\lambda| \leq \sum_{n=1}^{+\infty} \sum_{(x,I)\in\alpha_n\cap\pi} |g(x)||f(x)\lambda(I) - \int_I f d\lambda| \leq$$

$$\sum_{n=1}^{+\infty} n \sum_{(x,I)\in\alpha_n\cap\pi} |f(x)\lambda(I) - \int_I f d\lambda| \le \sum_{n=1}^{+\infty} n\frac{\varepsilon}{n2^n} = \varepsilon. \tag{70}$$

This implies that G and K are variationally equivalent, and the theorem follows.

3.4. Corollary. *Let $f \in \mathcal{H}$ on I_0 and $g : I_0 \to \mathbb{R}$. Then $fg \in \mathcal{H}$ if and only if g is Henstock-integrable with respect to $\tilde{f}$ and*

$$\int\int_{I_0} f(x, y)g(x, y)dxdy = \int\int_{I_0} g(x, y)d\tilde{f}(x, y). \tag{71}$$

Proof. Notice that if

$$H(I) = \int\int_I f(x, y)dxdy \tag{72}$$

then for $I = [a, b] \times [c, d]$

$$H(I) = \tilde{f}(b, d) - \tilde{f}(a, c) - \tilde{f}(b, c) + \tilde{f}(a, c), \tag{73}$$

so that

$$\int\int_{I_0} gdH = \int\int_{I_0} g(x, y)d\tilde{f}(x, y). \tag{74}$$

The rest follows now from theorem **3.3.**

3.5. Corollary. *If $f \in \mathcal{H}$ and $\mu \in \mathcal{M}$ then g_μ is Henstock integrable with respect to $\tilde{f}$ and*

$$\int\int_{I_0} g_\mu d\tilde{f}(x, y) = \tag{75}$$

$$\int\int_{I_0} \tilde{f}d\mu - \int_0^1 \tilde{f}(t, 1)dg_\mu(t, 1) - \int_0^1 \tilde{f}(1, t)dg_\mu(1, t) + \tilde{f}(1, 1)g_\mu(1, 1).$$

Proof. This is another form of the integration by parts formula and it follows directly from corollary 3.4 and proposition 2.14.

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Piotr Mikusiński, Department of Mathematics, University of Central Florida, FL 32816

Krzysztof Ostaszewski[1], Department of Mathematics, University of Louisville Louisville, KY 40292

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