Optimal Funding of a Liability

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Abstract: Funding of a given set of cash flow liabilities is typically arranged through level premium or a single premium. The question of optimality of the stream of premium payments has been largely ignored in the existing insurance literature. In this work, we propose a set of natural constraints on the premium flows and develop an optimal premium structure to fund any existing liability structure. The result shows how the least costly, yet actuarially sound, funding method—from the perspective of the insured or some more general paying agent—can be developed. The solution applies to either long-term or short-term models, but relies on deterministic assumptions, and thus is more likely to be applicable for shorter-term liabilities or those long-term liabilities whose cash flows can be determined with great certainty.

INTRODUCTION

Accumulation of financial assets in order to provide payments in case of some future events lies at the very core of the insurance business. Typically, insured events are uncertain. However, the insurance firm's existence relies always on some form of diversification of risk, through the application of the Central Limit Theorem, or the Law of Large Numbers, which makes the future payment less uncertain and in some cases completely deterministic. The insurance company receives premium payments to provide for such future payments. What we believe to be missing in actuarial and insurance literature is the question of what pattern of payments, from now until the future payout, is actually optimal, from the point of view of the insurance firm or its customers.

The pattern of accumulation of financial assets to provide for an existing liability is often a controversial issue in practice. The actuarial approach typically favors early accumulation of assets, as a matter of

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Journal of Insurance Issues, 2001, 24, 1 &2, pp. 17–29.

prudence, and for protection of the solvency of the firm. But if some of the assets can be used for payment of current—e.g., marketing—expenses, future growth opportunities can be expanded and the value of the firm may increase. This would suggest postponing at least some of the payments providing for future liability payment. In the case of an employer-sponsored pension plan, the employer may indeed be of the opinion that the funds would be better invested in the internal growth of the firm instead of financial assets, and postpone actual funding of pension liabilities until the very payment of them. Although in the United States such course of action is usually precluded by the provisions of the Employee Retirement Income Security Act of 1974, it is not unusual in Germany and Japan.

There has been a long-standing perception of conflict between the actuarial function within an insurance firm and the firm's other units. Chalke (1991) addressed this issue from the perspective of conflicting interests of the actuarial unit and the marketing unit, pointing out that one can seek to reconcile the two by trying to optimize the firm's overall profits in product design and marketing. The key value of Chalke's insight lies in bringing the economic perspective to actuarial work, thus pointing out that cost-plus pricing of insurance should be replaced by pricing practices that have replaced cost-plus in other industries, leading to more efficient allocation of capital (for the economic perspective on pricing, see, for example, Stigler, 1966).

The other insight of Chalke's work was its emphasis on optimization in product design. Let us note the obvious here by pointing out that the purpose of product design, premium calculation, and, later on, setting of the firm's reserves is to provide funding for liabilities created by risks assumed by the firm. While the question of the appropriateness of the funding level has received the highest level of attention in actuarial analysis, the question of optimality of the funding method has been largely ignored in insurance literature. Traditional insurance premiums, especially for long-term contracts, such as life insurance or annuities, are level. There have been innovations in this area with the introduction of single premium or flexible premium life and annuity products, but we should note that these new premium patterns are created for products for which the act of premium payment creates a liability, and not in the situation where liabilities are independent of premium payment. One classic example of such a situation of pre-determined liabilities is pension funding, and the question of optimality is not addressed in that context either (see, e.g., Aitken, 1996, and McGill et al., 1996).

In this work, we address the natural (in our opinion) question: what is the optimal pattern of premium payment to provide funding for an existing liability that is independent of the premiums? This question arises naturally in pension funding, but also in any other discharge of fixed liabilities. For example, many pension reforms of state pay-as-you-go systems worldwide involved some form of paying of accrued pension liabilities for which such state systems provide no assets. And existing pension-funding methods, while utilizing probabilistic methodology of life contingencies in theory, in practice rely on calculations involving expected values, in essence equivalent to purely deterministic models.

This work investigates what the most effective, yet still actuarially sound, method of achieving such funding is.

THE PROBLEM

In the simple accounting context, it has often been argued that when paying for a future liability, one should spread the payments over time, as this lowers risk. Let us examine the premise behind this belief. Assume that we consider a payment for the future liability to be a random game in which the objective is to have $P = E(X)$, where X is a random variable describing the outcome of the game. In this game, the risk can be described by its variance, $Var(X)$. Now suppose that instead we play n independent games with each played for (1/n)-th of the previous amount. Then the expected outcome of the game is still $E(X)$, but its variance is lowered to $\frac{1}{n}Var(X)$, thus significantly lowering the risk. This shows the logic behind the accounting perspective. However, such an approach does not ask what pattern of payments, whether level payments or some other arrangement, is optimal. We will turn to that question now.

We will assume for now that all cash flows are deterministic, and subject to a force of interest function $\{\delta_t : t \geq 0\}$. We will also assume that all cash flows occur in a finite time horizon, within a time interval $[0, T]$. We assume liability (possibly including expenses) cash flows C_t occur at discrete times $t = 1, 2, \ldots, T$ (some of the values of C_t could be zero). Their present values are: $L_t = C_t \exp\left(-\int_0^t \delta_s ds\right)$, $t = 1, 2, \ldots T$. Let us also denote the present value at time 0 of liabilities accrued through time t by $AL_t = \sum_{i=1}^{t} L_i$.

Let us also define $AL_0 = 0$. The liabilities as defined here are to be funded through payment of a series of premiums P_t, indexed by times $t = 1, 2, \ldots, T$ (some of the values of P_t could be zero). To follow the industry practice, we assume throughout the paper that the premiums are paid at the beginning of each time interval, but we will associate the premium paid at time $t-1$ with time t—i.e., first-year premium P_1 is paid at time 0, second-year premium P_2 is paid at time 1, etc. The present value of P_t is $PV(P_t) = P_t exp\left(\int_0^{t-1} \delta_s ds\right)$. We will also denote the present value of premiums paid through time t, but before it, by $AP_t = \sum_{i=1}^{t} PV(P_i)$.

As we have indicated above, our goal is to determine an optimal pattern for the premium payment flow. Determination of a proper method of funding of existing liabilities may indeed be considered to be at the very heart of the job of an actuary. One could choose a non-actuarial approach and postpone funding indefinitely into the future, while generating cash by creation of new liabilities (this could be termed the "Lucy Funding Method," modeled after the famous Lucille Ball line: "I can't be overdrawn, I still have some checks left"). Or we could choose the other extreme of a single premium P, paid at time $t = 0$. And then there is the traditional approach of level premiums. Instead, we postulate the following criteria for optimality of the funding method.

Constraint 1. The present value of premiums $PV(P_t)$ is a non-increasing function of time—i.e., $PV(P_t) \geq PV(P_{t+1})$ for $t = 1, 2, \ldots, T-1$.

This is, in our opinion, the key to the funding method meeting the test of actuarial prudence. While postponing of funding is allowed, premiums can, at most, be level in terms of their present values. Clearly, level funding, even at zero interest rate, meets this test. So does a single premium P_1 paid at time $t = 0$ (full pre-funding). On the other hand, the "Lucy Funding Method" fails this critical test.

Constraint 2. Accrued premiums at any point in time exceed accrued liabilities payable through that point in time—i.e., $AP_t \geq AL_t$ for $t = 1, 2, \ldots, T$, and $AP_T = AL_T$.

This means that the reserve is nonnegative at any time, and the liabilities are fully paid off at the end of the time horizon, while premiums are used solely for the purpose of fully discharging the liabilities (this does not necessarily imply that these are net premiums, as we possibly allow for inclusion of expenses and cost of capital among liabilities cash flows).

Under these two constraints, we seek to minimize $AP_t - AL_t$ for $t = 1, 2, \ldots, T$.

Note that we do not seek to minimize a single quantity, but rather to simultaneously minimize all possible reserves by choosing one of all possible sets of premiums $\{P_1, \ldots, P_T\}$ meeting Constraints 1 and 2. This minimization condition effectively calls for the smallest possible reserves throughout the existence of the policy, subject to Constraint 1 calling for not delaying the funding into the future, and Condition 2 calling for proper reserving. In other words, we seek to minimize the cost of funding while remaining actuarially sound. This addresses the needs of both the insured and the insurance firm.

Let us also observe that if we were to replace inequality in Constraint 1 by equality, this would, generally, lead to a contradiction with Constraint 2.

THE SOLUTION

Let F be a real valued piecewise linear continuous function defined on $[0, T]$ such that $F(0) = 0$. Then F is absolutely continuous and differentiable with the derivative f everywhere except for a finite number of points where f is not well defined. We can extend f on $[0, T]$ by assuming it to equal 0 at 0 and to be continuous from the left-hand side of $(0, T]$, thus defining uniquely the derivative of F, continuous from the left-hand side.

Obviously, for every $t \in [0, T]$, $F(t) = \int_0^t f(s)ds$.

The function $F(t) = AL_t$ is defined at all integral values of $t \in [0, T]$. Then F is a function of time representing present value of liabilities (e.g., pension liabilities) accrued through time t. Let us extend its definition to all $t \in [0, T]$ by making it piecewise linear and continuous, with a number of non-differentiability points as small as possible. Let F^* be the smallest concave majorant of F—i.e., F^* is the smallest real valued function defined on $[0, T]$ such that F^* is concave and $F^*(t) \geq F(t)$ for all $t \in [0, T]$. It is clear also that F^* could be defined as the smallest concave majorant for the function $F_1(t) = AL_t$ at all integral values of t, and $= 0$ otherwise on $[0, T]$. Let f^* be the continuous from the left-hand side derivative of F^* (recall that $f^*(0) = 0$ by definition). Then the premiums P_t^* minimize $AP_t - AL_t$ for $t = 1, 2, \ldots, T$ among all possible premiums satisfying Constraint 1 and Constraint 2, if and only if: $PV(P_t^*) = f^*(t)$ for $t = 1, 2, \ldots, T$.

Let us prove this assertion. First, let us consider an arbitrary premium flow $\{P_1, \ldots, P_T\}$ that meets Constraints 1 and 2, and minimizes $AP_t - AL_t$ for $t = 1, 2, \ldots, T$.

Because $PV(P_t)$ is a non-increasing function of t, the piecewise linear continuous function F^* defined by interpolating linearly the relationship $t \to AP_t$ to all $t \in [0, T]$ is concave. Constraint 2 implies that $F^*(t) \geq F(t)$ where $F(t) = AL_t$ for all $t = 1, 2, \ldots, T$. Because $AP_t - AL_t$ is minimized for $t = 1, 2, \ldots, T$, $F^*(t)$ is the smallest concave majorant.

On the other hand, if F^* is the smallest concave majorant for $F(t) = AL_t$ defined at all integral values of t, and then extended to all $t \in [0, T]$, then F^* satisfies Constraint 1 because it is a majorant, and Constraint 2 because it is concave, and it minimizes all $AP_t - AL_t$ for $t = 1, 2, \ldots, T$ because it is the smallest among concave majorants.

Determination of the premium by the method determined here is illustrated in Figure 1.

The premium flow defined here has one more important optimality property. Within the class of all premium flows $(P_1, \ldots, P_T)$ that satisfy Constraints 1 and 2, and also have the property that $AP_T = AL_T$ (premiums are used solely to fully discharge the liabilities), premiums $(P_1^*, \ldots, P_T^*)$ determined by the derivative of the smallest concave majorant give the minimum value of square error measure $\sum_{t=1}^{T} (Q - Q_t)^2$ where Q_t is the present value of P_t and $Q = \frac{1}{T}\sum_{t=1}^{T} L_t$ is the level funding in

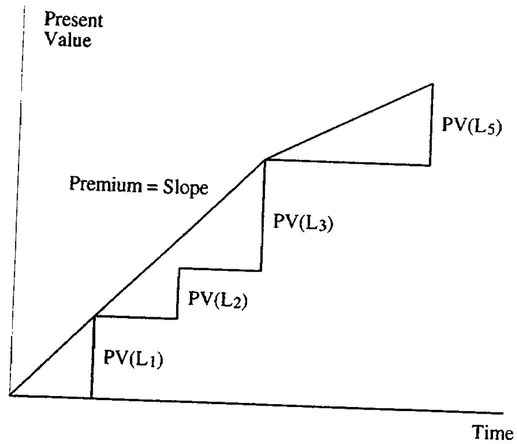

Fig. 1. Optimal Funding Given by the Smallest Concave Majorant

present value form. Let us prove this assertion. Assume $(P_1, \ldots, P_T)$ to be an arbitrary premium flow, and $(Q_1, \ldots, Q_T)$ be their present value. Define

$$E(Q_1, \ldots, Q_T) = \sum_{t=1}^{T} (Q_t - Q)^2 .$$

Note that the functional $E : \Re^T \to \Re_+$ is convex. Denote by D the set of all funding methods satisfying Constraints 1 and 2. Clearly D is a convex set. Let $Q_t^* = PV(P_t^*)$ for $t = 1, 2, \ldots, T$ (smallest concave majorant funding). It is easy to show that E achieves its minimum on D at the funding $(P_1^*, \ldots, P_T^*$ if for any other premium flow $(P_1, \ldots, P_T$ from D the following inequality holds: $\nabla E \lozenge (Q_1 - Q_1^*, \ldots, Q_T - Q_T^*) \geq 0$, where ∇E means the gradient of E at $(P_1^*, \ldots, P_T^*)$. This simply means that the directional derivative of E in the direction from $(Q_1^*, \ldots, Q_T^*)$ to $(Q_1, \ldots, Q_T)$ is nonnegative (or nonpositive in the opposite direction), and since $(Q_1^*, \ldots, Q_T^*)$ can be connected with a line segment with any other element of the convex set D, minimum must be achieved at $(Q_1^*, \ldots, Q_T^*)$, as desired. By calculating the gradient of E, the above inequality may be written as follows:

$$\sum_{t=1}^{T} (Q_t{}^* - Q)(Q_t - Q_t{}^*) \geq 0 .$$

Now we use the identity

$$\sum_{t=1}^{T} a_t b_t = a_T \sum_{t=1}^{T} b_t - \sum_{t=1}^{T-1} (a_{t+1} - a_t) \sum_{s=1}^{t} b_s ,$$

that holds for any sequences of real numbers $(a_1, \ldots, a_T)$ and $(b_1, \ldots, b_T)$. Setting $a_t = Q_t{}^* - Q$ and $b_t = Q_t - Q_t{}^*$ and applying the above-mentioned identity, we get

$$\sum_{t=1}^{T} (Q_t{}^* - Q)(Q_t - Q_t{}^*) =$$

$$(Q_t{}^* - Q)\left(\sum_{t=1}^{T} Q_t - \sum_{t=1}^{T} Q_t{}^*\right) - \sum_{t=1}^{T-1} (Q^*_{t+1} - Q_t{}^*) \sum_{s=1}^{t} (Q_s - Q_s{}^*) =$$

$$-\sum_{t=1}^{T-1} (Q^*_{t+1} - Q_t{}^*)\left(\sum_{s=1}^{t} Q_s - \sum_{s=1}^{t} Q_s{}^*\right) \geq 0$$

where the last inequality holds because $Q_t{}^*$ is a non-increasing function of time and $\sum_{s=1}^{t} Q_s{}^*$ defines the smallest concave majorant of accrued liabilities, hence $\sum_{s=1}^{t} Q_s$ cannot be smaller than $\sum_{s=1}^{t} Q_s{}^*$. Observe that the property proven above means also that the optimal funding $(P_1{}^*, \ldots, P_T{}^*)$ has the smallest possible volatility measured by the variance of the present values of premiums, under prescribed constraints.

While the solution presented here is specific to a prescribed interest rate scenario δ_t, it allows for an arbitrary scenario with nonnegative interest rates (negative interest rates cause level premiums to violate Constraint 1, and we consider this extremely undesirable and unrealistic, so we do not allow negative interest rates here). It also requires deterministic cash flows within that scenario.

The procedure can also allow for an inclusion of the "cost of capital" margin to premiums, in two possible ways:

(i) by considering "cost of capital" payments a separate expense, earning a different, arguably higher, interest rate;

(ii) by assuming that capital earns the same interest rate as the other cash flows, while minimizing the impact of margin inclusion on the premium level.

Let us discuss these propositions. Approach (i) treats each payment of a return to capital as an expense in addition to liabilities and expense payments already accounted for. However, these returns to capital are added to liabilities payments only after their present values are evaluated. Such present values must be calculated using a different discount rate, a higher one, and appropriately set to the insurance firm's cost of capital. Because the optimal set of premiums is determined by the set of present values of liabilities, expenses, and/or capital cash flows, the fact that some of those may be calculated using different discount functions does not alter the optimization process.

Under the second approach (ii), we begin by finding the accrued value of the capital invested at the end of the period, call it AC. Then we find the maximum value of a period premium, α. Note that α is also equal to the maximum slope of the smallest concave majorant, and because of the concavity, α is the slope of the first segment of the smallest concave majorant. If $\alpha T \leq AL_T + AC$, then even the largest periodic premium paid as a level premium is not sufficient to cover the additional cost of capital. In this case, a level premium paying for liabilities, expenses, and capital is the optimal one. If $\alpha T \geq AL_T + AC$, then the optimal premium is obtained by increasing each premium following consecutively, to the level of α, followed by a smallest concave majorant, until the premium flow is sufficient to pay for $AL_T + AC$.

We should note that this procedure results in the earliest possible return of capital, given the rate of return required on capital. Indeed, capital payments can be accumulated and discounted by a higher rate, the firm's cost of capital, and then the procedure can be applied to the resulting values of capital cash flows. The key difference between this procedure and method (i) is the fact that (i) assumes return of capital cash flows to be given, while (ii) brings them as far forward as possible without disturbing the existing premium flow, in particular without raising the first premium.

One more issue should be raised here. In practice, one of the major problems for funding of insured cash flows is the fact that insurers are required to meet reserves in early policy years when expenses tend to deplete the premiums generated from the contract. This means that early cash flows for benefits and expenses tend to be substantial in relation to later cash flows. The funding method defined by the derivative of the smallest concave majorant would actually call for higher premiums in the early policy years, and imply pay-as-you-go premium for those years. This represents a major departure from the common practice of level premiums, but may be an issue to consider by companies and regulators dealing with this important practical problem.

STOCHASTIC INTEREST RATES

The optimal procedure proposed here applies to a given set of present values of liabilities, expenses, and capital cash flows. But the present values up to now have been assumed to be calculated using deterministic interest rates. This, of course, is unrealistic. Future interest rates are uncertain. Optimality conditions developed here do not overcome the problem of interest rate risk going forward.

The Fundamental Theorem of Asset Pricing (Ross, 1976; Dybvig and Ross, 1989) states that in an efficient market, absence of arbitrage is equivalent to prices of capital assets being given as expected present values of their cash flows in a risk-neutral world. In practice of valuation of liabilities, this is typically addressed by discounting liabilities cash flows under various interest-rate scenarios, and then taking the price to be the expected present value from the distribution of present values under various scenarios. Our optimal premium procedure produces a unique solution for the set of present values of liabilities, expenses, and capital cash flows under each scenario. How does one proceed from this to the global solution?

If the interest-rate scenarios produced indeed come from a risk-neutral world, it is clear that the expected value of the present value of liabilities, expenses, and capital cash flow is indeed the actual price, or market value. Therefore, our procedure can be naturally applied to these expected present values, and the set of optimal present values can be produced. But the procedure requires that timing of liabilities cash flows be set. The optionality of most liabilities cash flows expresses itself mostly in uncertainty of timing of cash flows. Timing is, in fact, assured only under each specific interest-rate scenario. That makes this simple solution unrealistic.

One can reach for the second-best, solution however. It is clear that an optimal premium flow can be produced for each interest-rate scenario. Each such optimum is given by a smallest concave majorant. But each is also a set of present values of premiums to be paid, thus representing a set of cash flows of a security in the financial marketplace. The expected present value is the price of that security, and we can create a new concave function by taking the expected value of all smallest concave majorants. The function so produced will, obviously, be concave. However, given that the timing of liabilities cash flows is uncertain, one cannot establish that domination of accrued premiums over accrued liabilities at every point in time, only at the beginning of the period. Indeed, this is the central problem of asset-liability management, and it does require further investigation.

We should note, however, that in many practical analyses of stochastic interest-rate scenarios, actuaries do analyze specific scenarios that produce the best and worst cases, from the point of view of solvency of the firm. This method would allow such practical analyses to be enhanced by adding information on the best funding method under the best case and under the worst case—such analyses currently missing in applicable actuarial work.

CONCLUSIONS AND COMMENTS

The result of this work sheds some new light on actuarial funding methodologies.

First, it allows for a reconsideration of traditional level funding perspective on life insurance premiums. If one can project liabilities deterministically—e.g., in a deterministic cohort—one can actually design optimal premium flow. The controversial part of this optimal solution is the fact that higher expenses in early years should be accompanied by higher premiums.

Second, it may change the pension funding perspective. Pension funding methods either call for separate, typically level, funding of unfunded previous service accrued liabilities, and of plan unfunded liabilities, or combine the two liabilities and search for level funding of them. In contrast, this method suggests first netting existing accrued liabilities' cash flows with existing asset cash flows and then the method calls for funding given by the derivative of the smallest concave majorant of the remaining liabilities.

Last but not least is the inference regarding social insurance. Social insurance is typically funded on a pay-as-you-go basis, but demographic variations can cause deviations from that principle, because of concern for funding retirement benefits for large generations. In fact, both the United States and Canada have departed from pay-as-you-go precisely for that reason. In the United States, such departure was introduced in 1983, when accumulation of the Trust Fund, to be discharged during the retirement of baby boomers, started. In Canada recently, so-called "smart funding" was introduced. It calls for greater accumulation of a fund during periods of relatively high interest rates and its discharge during periods of lower interest rates. Underlying economic theory for such an approach can be twofold:

- Investment perspective generally implies mean reversion in rates of return, thus high rates should be followed by lower rates, and vice versa. It is, of course, difficult to decide exactly what constitutes "high" and "low" rates, as a 7 percent long-term Government Bond was perceived as a high rate in 1960s and as a low rate in most of 1980s and 1990s.
- Higher payroll taxes under higher interest rates, and lower payroll taxes under lower interest rates, might provide a desirable counter-cyclical macroeconomic effect.

This smallest concave majorant method provides an interesting alternative, also producing fund accumulation, but on a relatively lower scale. Policymakers may want to consider this perspective.

We believe this funding method may indeed prove useful in funding various forms of liabilities in a new fashion, preserving actuarial prudence, while allowing premiums that conform to the structure of liabilities. It should also be noted that the question of optimal funding of random future liabilities, while not addressed in our work, is a very promising area of future research suggested by our work.

NOTE

[1] The authors have received research support from the Scientific Research Committee of the Republic of Poland, Grant 1H02B01814.

REFERENCES

Aitken, William (1996) *Pension Funding and Valuation: Problem Solving Approach,* 2nd Edition. Winsted, CT: Actex Publications.

Black, Kenneth Jr., and Harold Skipper Jr., (1994) *Life Insurance,* 12th Edition. Upper Saddle River, NJ: Prentice-Hall.

Bowers, Newton, et al. (1996) *Actuarial Mathematics,* 2nd Edition, Schaumburg, Illinois: Society of Actuaries.

Chalke, Shane A. (1991) "Macro Pricing: A Comprehensive Product Development Process," *Transactions of the Society of Actuaries,* 43, pp. 137–194, Discussions pp. 195–230.

Dybvig, P.H. and Stephen A. Ross (1989) "Arbitrage," in *The New Palgrave: A Dictionary of Economics,* vol. 1, edited by J. Eatwell, M. Milgate, and P. Newman. London: Macmillan, 1987, pp. 100–106. Reprinted in *The New Palgrave: Finance,* edited by J. Eatwell, M. Milgate, and P. Newman. New York: W.W. Norton, 1989, pp. 57–71.

McGill, Ben, Kyle Brown, John Haley, and Sylvester Schieber (1996) *Fundamentals of Private Pensions*, 7th Edition, Philadelphia, PA: University of Pennsylvania Press.

Ross, Stephen (1976) "The Arbitrage Theory of Capital Asset Pricing," *Journal of Economic Theory*, 13, pp. 341–360.

Rudin, Walter (1987) *Real and Complex Analysis*, 3rd Edition. New York: McGraw-Hill.

Stigler, George (1966) *The Theory of Price*. New York: Macmillan.