Pricing Life Insurance: Combining Economic, Financial, and Actuarial Approaches

Hong Mao, James M. Carson, Krzysztof M. Ostaszewski, and Luo Shoucheng*

Abstract: This paper examines the pricing of term life insurance based on the economic approach of profit maximization, and incorporating the financial approach of stochastic interest rates, investment returns, and the insolvency option, while also including actuarial modeling of mortality risk. Optimal price (premium) is obtained by optimizing a stochastic objective function based on maximizing the expected net present value (NPV) of insurer profit. Expected claim payments are calculated on the basis of the Cox, Ingersoll, Ross (1985) financial valuation model. Our work analyzes numerically the influence of various parameters on optimal price, optimal expected NPV of insurer profit, and the insolvency put option value. We examine several parameters including the speed of adjustment in the mean reverting prices, the initial value of the short run equilibrium interest rate, the volatility of interest rate, the volatility of asset portfolio, the long run equilibrium interest rate, and the age of the insured. Findings demonstrate that optimal prices generally are most sensitive to changes in the long run equilibrium interest rate. Factors that have a strong influence on the price of the insolvency option include the age of the insured, volatility of interest rate, and volatility of the asset portfolio, especially at larger values of these parameters. [Key words: life insurance pricing; economic pricing; financial pricing; actuarial pricing; stochastic optimization; insolvency]

* Hong Mao (hmaoi@online.sh.cn) is Associate Professor of Insurance at Shanghai Second Polytechnic University. James M. Carson (jcarson@cob.fsu.edu) holds the Midyette Eminent Scholar Chair of Risk and Insurance in the College of Business at Florida State University. Krzysztof M. Ostaszewski (krzysio@ilstu.edu) is Professor and Director of the Actuarial Science Program at Illinois State University. Shoucheng Luo (luo@sh163.net) is Associate Professor of Management at Shanghai Second Polytechnic University. The authors thank participants at a meeting of the American Risk and Insurance Association for helpful comments. Hong and Luo are grateful for support provided by the China-Canada University Partnership Program—National Science Foundation of China (No. 70142014) and by Shanghai Municipal Arts and Social Science Fund (No. 010W01).

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INTRODUCTION

Early life insurance pricing models generally followed one of two paths: a focus on life risks with little attention to other aspects (e.g., investment risks), or a focus on financial valuation principles with little attention to the insurance liabilities side. Over time, pricing models have evolved to include both underwriting and investment risks. Such models are reviewed by Cummins (1991). From the 1970s through the 1990s, research examined insurance pricing in competitive markets. For example, Spellmam, Witt, and Rentz (1975) developed an insurance pricing method based on microeconomic theory in which investment income and the effect of the elasticity of demand are considered, and price is determined by maximizing profit. McCabe and Witt (1980) discussed insurance prices and regulation under uncertainty. They considered underwriting and investment risks, as well as the cost of regulation imposed on the insurer. In addition, demand for the insurer's product was assumed to be a function of the insurance rate and the average time the insurer takes to settle claims. More recently, Persson and Aase (1997) developed a model for pricing life insurance that includes a guaranteed minimum return under stochastic interest rates. In their pricing model, investment and mortality risks are considered simultaneously.

Cummins and Danzon (1997) developed a two-period pricing model subject to default risk. Demand for insurance is inversely related to insurance default risk and is imperfectly price elastic. Kliger and Levikson (1998) discuss pricing of short-term insurance contracts based on economic and probabilistic arguments. Their objective function in the maximization problem is defined as expected net profit, the loss resulting from insolvency, and the demand for insurance embedded in the objective function. Price and the number of insurance policies are determined by optimizing an objective function. Wang (2000) introduces a class of distortion operators for pricing financial and insurance risks. Schweizer (2001) combines insurance and financial research by embedding an actuarial valuation principle in a financial environment. Still other research addresses insurance pricing in competitive markets for property and liability insurance with one or two period cash flows. Related articles include Joskow (1973), D'Arcy and Garven (1990), Brockett and Witt (1991), Greg (1995), Sommer (1996), De Vylder (1997), Wang, Young, and Panjer (1997), Lai, Witt, Fung, MacMinn, and Brockett (2000), Gajek and Ostaszewski (2001), and Oh and Kang (2004).

The typical approach in life insurance is to model interest rates by a stochastic process (see Persson, 1998; Bacinello and Persson, 2002) and to derive price (premium) according to the equivalence principle (Bowers et al., chapter 6, 1997). The risks associated with interest rates and mortality typically are salient factors considered in establishing pricing models. Pricing methods based on a security loading factor that aims to achieve a desired low probability of insolvency assume that premiums are independent of the number of insureds. However, assuming at least some price elasticity of demand (Pindyck and Rubinfeld, 1998), such an assumption is inconsistent with the laws of supply and demand.

The goal of this paper is to extend previous research with multi-period life insurance pricing models that combine economic and actuarial criteria to maximize the expected net present value of insurer profit. In this study, the influence of interest rate risk, insolvency, and supply/demand are considered explicitly in optimal control models. Optimal prices (from insurer's perspective) are obtained by solving objective functions based on optimization techniques and Monte Carlo simulation. The effects of various parameters (interest rates, volatility, and age) on optimal prices, optimal expected NPVs of insurer profit, and values of the insolvency put option are illustrated with numerical examples. The next section discusses life insurance pricing models.

PRICING MODELS

Assumptions Underlying the Pricing Models

In this section, we discuss the assumptions underlying the development of the pricing models used in this study. The insurer sells only life insurance. We consider single-premium and level-premium term life insurance policies with term Y. (Note that the concept of *single premium term life insurance* may seem foreign to some readers, especially since term life is not marketed this way; however, this approach is standard in actuarial pricing, as in Bowers et al., 1997.)

In the stochastic control model (see Ferguson and Lim, 1998) described below, the contract is a contingent-claim affected by both mortality and financial risk. Stochastic interest rates are used as discount rates that are treated as a continuous time-stochastic process where mortality also is considered as a random component (see Giaccotto, 1986; and Panjer and Bellhouse, 1980). The single premium and level premium models treat the insurance policyholder as analogous to an investor buying a financial asset. The insurer raises funds from policyholders, invests the funds, and pays benefits including investment income when claims occur.

Optimal price levels of the insurer's life insurance products are postulated to be dependent on the insurer's claims, non-claim expense, financial strength (solvency), and the market demand. Moreover, price is considered demand-elastic with endogenous insolvency risks, where interest rates and mortality are independent of each other. Also, the insurer is assumed to be risk neutral, where price is set according to the expectation criterion; that is, the objective of the insurance company is to maximize the expected net present value of insurer profit—the difference between the expected present value of income and the expected present value of payments.

The pricing models do not impose binding constraints from rate regulation. However, insolvency occurs if the insurer's wealth decreases beyond a minimum reserve required by law. The firm is assumed to have market power and can vary its premium volume by varying price (i.e., we do not assume perfectly competitive insurance markets, but we note that decreasing demand function also applies to all companies in a competitive industry). Financial markets are assumed to be perfectly competitive, frictionless, and free of arbitrage opportunities. All consumers purchase the same unit of insurance coverage, and market demand is a function of price, age of the insured, maturity time of insurance contracts, and default risk. Moreover, for modeling purposes, all policyholders are assumed to be rational and non-satiated, and to share the same information.

For both single premium and level premium models, a closed–form solution for the default-free discount bond price is desirable. Several models have been developed to calculate prices of default-free discount bonds.[1] Here we will employ the Cox, Ingersoll, and Ross (1985) model. This model describes the valuation of a zero-coupon bond. The model specifies that the short-term interest rate, r, follows an Ornstein-Uhlenbeck mean reverting stochastic process. Specifically, $dr = \kappa(\mu - r)dt + \sigma\sqrt{r}dz$, where z denotes a standard Wiener process, σ denotes the volatility of interest rates, μ is the long run equilibrium interest rate, the gap between its long run equilibrium and current level is represented by $\mu - r$, and κ is a measure of the sense of urgency exhibited in financial markets to close the gap and gives the speed at which the gap is reduced, where the speed is expressed in annual terms. Let $P(r, t, Y)$ express the discount value of a zero-coupon bond, so that the valuation equation is $\frac{1}{2}\sigma^2 rP_{rr} + k(\mu - r)P_r - P_\tau - rP = 0$, where subscripts on P denote partial derivatives and $\tau = Y - t$, where Y is the maturity time.

As solved by Cox, Ingersoll, and Ross, the price of a zero-coupon bond is given by:

$$P(r, \tau) = A(\tau)e^{-B(\tau)r}$$

$$\text{where } A(\tau) = \left[\frac{2\gamma e^{(\kappa+\gamma)\tau/2}}{g(\tau)}\right]^{2\kappa\mu/\sigma^2}$$

$$B(\tau) = \frac{2(e^{\gamma\tau}-1)}{g(\tau)} \tag{1}$$

$$g(\tau) = 2\gamma + (\kappa+\gamma)(e^{\gamma\tau}-1)$$

$$\gamma = \sqrt{\kappa^2 + 2\sigma^2}\,.$$

The above result will be used in the analysis that follows.

Single Premium Term Life Insurance Contracts

For single premium term insurance policies, the expected net present value of the policy's cash flows is *ENPV(n)*. The expected net present value of the policy cash flows equals the difference between the expected present value of income and the expected present value of payments.

The objective function satisfies the constraint that the market price (premium) is positive and is defined as:

$$\text{Max } ENPV(n) = PI(n) - PL(n) \tag{2}$$

$$\text{Subject to } PP(n, b(n), \pi^1(x, Y)) > 0\,.$$

The Present Value of Income

Assume that the insurance firm faces a price for policies that depends on quantities of policies, insolvency risks (financial strength), and claim payment: $PP(n, b(n), \pi^1(x, Y))$, where n equals the quantity of insurance sold, $\pi^1(x, Y)$ equals the expected present value of claim payment for each exposure unit, x equals the age of the insured, Y equals the maturity time of insurance contracts, and *b(n)* equals the value of the insolvency put option—the current value of the owners' option to default if liabilities exceed assets at the claim payment date. The value of the insolvency put option is inversely related to the price and liability.[2]

Therefore, the present value of income is:

$$PI(n) = PP(n, b(n), \pi^1(x, Y))n\,.$$

Table 1 shows the notation that is used in the pricing models.

Table 1. Notation Used for Pricing Models

$b(n)$	the value of insolvency put option for single premium life policy
n	quantity of insurance sold for single premium life policy
$PP(n, b(n), \pi^1(x, Y))$	a single premium rate based on market price
$T = T(x)$	the remaining life time of an x year old insured
Y	the maturity time of insurance contracts
$f_x(t)$	the probability density function of T
${}_tp_x$	the survival probability
$C(t)$	benefit payable (or claim payment) upon death at time t
p	the non-claim payment expense percentage of claim payment
π^1	expected present value of claim payment for each exposure unit
PI	expected present value of income for a single premium life policy
PL	expected present value of claim payment for a single premium life policy
$ENPV(n)$	expected net present value of a single premium life policy = $PI(n) - PL(n)$
AA	constant of the demand function (assumed linear) for single premium life policy
A_1	constant of demand equation for level premium life policy
B,G,F	coefficients of the demand function for single premium life policy corresponding to quantity of demand, insolvency risk, and age of the insured, respectively
μ	long run equilibrium interest level (assumed independent of t)
σ_1	standard deviation of cumulative investment in widely diversified portfolio (assumed independent of t)
σ	standard deviation of interest rate (assumed independent of t)
r	short run interest rate
κ	the speed of adjustment in the mean reverting prices

Note: In the analysis that follows, the appropriate terms are subscripted with "1" for level premium analysis.

The Present Value of Payments

Let *C(t)* denote an arbitrary insurance benefit payable at time *t*. Let the random variable *T* = *T(x)* denote the remaining lifetime of an *x*-year-old insured, and let ${}_tp_x = P(T > t)$ denote the probability that the insured survives to age $x + t$ given that he is alive at age *x* and ${}_tq_x = 1 - {}_tp_x$ denote the probability that the insured will die between *x* and $x + t$ given that he is alive at age *x*. We assume that the probability density function of *T* exists and denote it by $f_x(\bullet)$.

Assume that *T* and the pair ($C(\bullet)$ and $v(\bullet)$) are independent random variables. Let us represent the non-claim payment expenses as *p* percent of claim payments for *n* policies. For simplicity, taxes are ignored here. The quantity $v(t) = exp\left(-\int_0^t r_u du\right)$ is the discount function and represents the present value at time zero of one unit of account at time *t* discounted by the stochastic interest rate. Then the expected present value of benefit *C(t)* (claim payment) of term life insurance, payable upon death at time $t \le Y$, is

$$\pi^1(x, Y) = \int_0^Y E[v(t)C(t)]f_x(t)dt, \tag{3}$$

where $E(\bullet)$ denotes the expected value operator. Note that equation (3) includes product of mortality factor and the expected value of pure financial claims.

Using the Cox, Ingersoll, Ross closed form of valuation and letting *C(t)* = 1, then the expected present value of benefit (claim payment) payable upon death at time *t* is

$$\begin{aligned}\pi^1(x, Y) &= \int_0^Y E\left(\exp\left(-\int_0^t r_u du\right)\right)f_x(t)dt \\ &= \int_0^Y P(r, t, T)f_x(t)dt \\ &= \int_0^Y A(t)e^{-B(t)r}f_x(t)dt \end{aligned}\tag{4}$$

$$\text{where } A(t) = \left[\frac{2\gamma e^{(\kappa+\gamma)t/2}}{g(t)}\right]^{2\kappa\mu/\sigma^2}$$

$$B(t) = \frac{2(e^{\gamma t}-1)}{g(t)}$$

$$g(t) = 2\gamma + (\kappa+\gamma)(e^{\gamma t}-1)$$

$$\gamma = \sqrt{\kappa^2 + 2\sigma^2}.$$

The Value of the Insolvency Put Option

The option pricing model has been applied to insurance pricing by several authors (see Cummins, 1991). Further literature includes Pennacchi (1999), Grosen and Jorgensen (2000), Milevsky and Posner (2001), Tan and Hu (2002), and Bacinello and Persson (2002). In this article, we extend the research to price insolvency risks that incorporate stochastic rates and also consider correlation between two stochastic processes of interest rates and accumulated investment.

Suppose that the initial asset D_0 is the premium income of the insurer where $D_0 = PP(n, b(n), \pi^1(x, Y))n$, which is deemed to be invested in a widely diversified portfolio. Denoting by D_t the market value at time t of the accumulated investment, the value of the insolvency put option at time t is $H_t = \max(X_t - D_t, 0)$ with exercise price X_t, where X_t is the cash flow of liability at time t and

$$X_t = {}_tq_x n(1+p). \tag{5}$$

Here we assume that the death benefit paid on each contract is one dollar.

Furthermore, we assume D_t is described by the following stochastic differential equation under the equivalent martingale measure: $dD_t = rD_t dt + \sigma_1 D_t dw^1$, where r satisfies the stochastic differential equation $dr = \kappa(\mu - r)dt + \sigma\sqrt{r}dw^2$, dw^1, dw^2 are correlated Wiener processes, and let $\rho_{1,2}$ express their instantaneous correlation coefficient.

Therefore, based on the definition of the insolvency put option and the equations of (1) and (5), the current value of the insolvency put option is equal to the aggregate discounted value of all individual put options with exercise price $X_t = {}_tq_x n(1+p), 0 \le t \le Y$; that is:

$$b(n) = E^Q\left[\int_0^Y v(t)dH_t\right]$$

$$= E^Q\left[\int_0^Y P(r,t)d\max(X_t - D_t, 0)\right], \tag{6}$$

$$= E^Q\left[\int_0^Y A(t)e^{-B(t)r}d\max({}_tq_x n(1+p) - D_t, 0)\right]$$

where $E^Q[\cdot]$ denotes the expectation operator under the equivalent martingale measure. By using numerical approximating algorithms, we can get the approximating solution of *b(n)* (see Appendix 1). For proof of existence of differentiation—$d\max({}_tq_x n(1+p) - D_t, 0)$—and continuity of function *b(n)*, please see Appendix 2.

Babbel, Jeremy, and Merrill (2002) discussed the fair value of liabilities in terms of financial economics. They indicate that the fair value of liabilities should equal the discounted liability cash flows minus the value of the insolvency put option, as there must be some accounting for risk.

Therefore, the expected present value of liability payments can be expressed as the difference between the expected present value of explicit payments (including claims and non-claim expenses) and the value of the insolvency put option. That is:

$$PL(n) = \pi^1(x, Y)(1+p)n - b(n), \tag{7}$$

where *b(n)* satisfies equation (6), so that the expected net present value of insurer profits is:

$$ENPV(n) = PI(n) - PL(n)$$

$$= PP(n, b(n), \pi^1(x, Y))n - \pi^1(x, Y)(1+p)n + b(n). \tag{8}$$

The optimization problem is:

$$\text{Max } ENPV(n) = PI(n) - PL(n)$$

$$= PP(n, b(n), \pi^1(x, Y))n - \pi^1(x, Y)(1 + p)n + b(n) \tag{9}$$

$$\text{Subject to } PP(n, b(n), \pi^1(x, Y)) > 0\,.$$

Given the parameters of μ, κ, σ_1, σ_2, Y, p, x, $\rho_{1,2}$, the initial value of r_0, the probability density of the remaining lifetime of an x-year-old insured, $f_x(\bullet)$, and the function of demand, then the optimal solutions consisting of n^*, $PP(n^*, b(n^*), \pi^1(x,Y)$, $b(n^*)$ and $ENPV(n^*)$ can be obtained by optimizing the objective function. Before presenting results for single premium term life contracts, we will examine the pricing of level premium term life contracts.

Level Premium Term Life Insurance Contracts

As noted in Table 1, for level premium term insurance policies, the subscript "l" is added to each term where appropriate. Consider a contract in which a periodical premium is paid at the beginning of each year, if the insured is alive. As in the case of a single premium term life insurance contract, the objective function can be described as:

$$\text{Max } ENPV_1(n_1) = PI_1(n_1) - PL_1(n_1)$$

$$\text{Subject to } PP_1(n_1, b_1(n_1), \pi^1(x, Y)) > 0\,. \tag{10}$$

Assume that the level premium based on market price is expressed as $PP_1(n_1, b_1(n_1), \pi^1(x, Y))$, and premium income occurs at the beginning of each year, so the present value of premium income is

$$PI_1(n) = \sum_{t=0}^{Y-1} PP_1(n_1, b_1(n_1), \pi^1(x, Y))\,{}_tp_x n_1 E\left(exp\left(-\int_0^t r_u du\right)\right)$$

$$= \sum_{t=0}^{Y-1} PP_1(n_1, b_1(n_1), \pi^1(x, Y))\,{}_tp_x n_1 P(r, t)$$

$$\text{where } P(r, t) = A(t)e^{-B(t)r}$$

$$A(t) = \left[\frac{2\gamma e^{(\kappa+\gamma)t/2}}{g(t)}\right]^{2\kappa\mu/\sigma^2} \tag{11}$$

$$B(t) = \frac{2(e^{\gamma t} - 1)}{g(t)}$$

$$g(t) = 2\gamma + (\kappa + \gamma)(e^{\gamma t} - 1)$$

$$\gamma = \sqrt{\kappa^2 + 2\sigma^2}.$$

We assume that the premium income at the beginning of each year is invested in a widely diversified portfolio. Let D_{jt}^1 express the market value of the cumulative investment fund at time t with initial value of $D_{j0}^1 = PP_1(n_1, b_1(n), \pi^1(x, Y))_j p_x n_1$, where D_{j0}^1 indicates the premium paid by the surviving policyholders at time *j*, *j* = 0, 1, 2 ...$Y - 1$, $t > j$. Therefore, the total market value of the cumulative investment fund at time t is $D_t^1 = \sum_{j=0}^{Y-1} D_{jt}^1$, where D_{jt}^1 satisfies the stochastic differential equation $dD_{jt}^1 = rD_{jt}^1 dt + \sigma_{j1}^1 D_{jt}^1 dw_j^1$, $j = 0, 1, 2, \ldots Y - 1$, $t > j$, r satisfies stochastic differential equation $dr = \kappa(\mu - r)dt + \sigma\sqrt{r}dw$, and $w, w_0^1, w_1^1, w_2^1 \ldots w_{\gamma-1}^1$ are $Y + 1$ Wiener processes, and let $\rho_{i,k}^1$ ($i, k = 0, 1, 2\ldots, Y$) express their instantaneous correlation coefficients.

Similarly as in the case of a single premium term life contract, the current value of the insolvency put option is:

$$b_1(n_1) = E^Q\left[\int_0^Y A(t)e^{-B(t)r} d\max({}_t q_x n_1(1 + p) - D_t^1, 0)\right]. \qquad (12)$$

As in the case of single premium life policies, the expected present value of payment $PL_1(n)$ is equal to:

$$\pi^1(x, Y)(1 + p)n_1 - b_1(n_1) \qquad (13)$$

where $b_1(n_1)$ satisfies equation (12).

Therefore, the objective function is:

$$\max ENPV_1(n_1) = PI_1(n_1) - PL_1(n_1)$$

$$= \sum_{t=0}^{T-1} PP_1(n_1, b_1(n_1), \pi^1(x, Y))_t p_x n_1 P(r, t) \qquad (14)$$

$$-(\pi^1(x, Y)(1+p)n_1 + b_1(n_1))$$

$$\text{Subject to } PP_1(n_1, b_1(n_1), \pi^1(x, Y)) > 0 .$$

Similarly as in the case of a single premium life policy, Monte Carlo simulation and optimization technique (written by the authors in Matlab 6.0), the optimum solutions of $n_1{}^*$, $P_1(n_1{}^* b_1(n_1{}^*), \pi^1(x,Y))$, $ENPV_1(n_1{}^*)$, *and* $b_1(n_1{}^*)$ are obtained. The next section discusses the numerical results and the sensitivity analysis.

NUMERICAL RESULTS AND SENSITIVITY ANALYSIS

First, let us illustrate the process by an example. For conciseness of exposition, we discuss only a single premium term life policy case. Consider the demand function:

$$PP(n, b(n), \pi^1(x, Y)) = AA - Bn - Gb(n) + F\pi^1(x, Y) . \qquad (15)$$

The rationality of this consideration is discussed here. First, demand theory suggests that marginal utility decreases with consumption, and here we assume demand is a linear decreasing function of price. Second, in the early 1990s a significant number of U.S. life insurance companies were unable to meet their obligations and became insolvent (e.g., see Browne et al., 1999), so the financial strength of insurance companies has become more of a salient issue in the life insurance transaction. Third, it is clear that price must be positively related to the expected present value of claim payment for each exposure unit ($\pi^1(x, Y)$), which is an increasing function of the initial age of the insured (x) and the maturity time of insurance contract (Y). Based on optimization techniques, Monte Carlo simulation in Matlab 6.0, and assumed values (as noted in Table 2), optimal solutions are shown in Table 2.

Table 2. Optimal Results with and without Insolvency Risk

$\kappa = 0.24, r_0 = 0.07, \mu = 0.05, \sigma = 0.1$				
	$\sigma_1 = 0.03$		$\sigma_1 = 0.5$	
	Optimum solutions without consideration of insolvency risk	Optimum solutions considering insolvency risk	Optimum solutions without consideration of insolvency risk	Optimum solutions considering insolvency risk
Max$ENVP(n^*)$	212.4891	212.4891	212.4891	111.9641
n^*	10308	10308	10308	4084
$PP(n^*,b(n^*), \pi^1(x,Y))$	0.0574	0.0574	0.0574	0.0552
$b(n^*)$	–	0	–	36.6580

Assumptions for other parameters:
$AA = 0.075$, $B = 2 \times 10^{-6}$, $G = 0.0004$, $F = 0.1$, $p = 0.2$, $x = 49$, $Y = 5$, $dt = 1$, $\rho_{1,2} = 0.5$, $q_{49} = 0.00612$, $q_{50} = 0.00663$, $q_{51} = 0.00720$, $q_{52} = 0.00784$, $q_{53} = 0.00857$

Table 2 illustrates the observation that when the long run equilibrium interest rate (μ), initial value of short run interest rate (r_0), the speed of adjustment in the mean reverting prices (κ), and the standard deviation of interest rates (σ) remain constant, and the standard deviation of accumulated investment (σ_1) increases, the risk of insolvency will strongly affect the optimal level of number of policies and prices. Therefore, it is shown that default risk is an important consideration when determining the solutions to the optimal number of policies and prices, especially in the case that σ_1 takes on larger values.

In addition, it should be noted that when the insolvency risk is not considered (let $b(n) = 0$), an explicit solution can be found simply by solving the partial differential equation of $\frac{\partial(ENPV(n):b(n) = 0)}{\partial n} = 0\left(n < \frac{AA + F\pi^1(x, Y)}{B}\right)$. The optimal solution is:

$$n^* = \begin{cases} \dfrac{AA + \pi^1(x, Y)(F - 1 - p)}{2B} & \text{when } AA < \pi^1(x, Y)(F + 1 + p), \\ — & \text{otherwise.} \end{cases} \tag{16}$$

By combining equation (16) with (8) and (15), and letting $b(n) = 0$, we can get the optimum solutions

(17)

$$PP(n^*, \pi^1(x, Y)) = \begin{cases} \dfrac{AA + \pi^1(x, Y)(F + 1 + p)}{2} & \text{when } AA < \pi^1(x, Y)(F + 1 + p), \\ \text{—} & \text{otherwise.} \end{cases}$$

(18)

$$ENPV(n^*) = \begin{cases} \dfrac{[AA + \pi^1(x, Y)(F - 1 - p)]^2}{4B} & \text{when } AA < \pi^1(x, Y)(F + 1 + p), \\ \text{—} & \text{otherwise.} \end{cases}$$

In the case that $AA = 0.075$, $F = 0.1$, $B = 2 \times 10^{-6}$, $\pi^1 = 0.0307$, $p = 0.20$, the explicit solutions are $PP(n^*, \pi^1(x, Y)) = 0.0574$, $ENPV(n^*) = 212.4891$, $n^* = 10308$, consistent with the solutions obtained by optimization techniques shown in Table 2.

The complexity of the optimization problem, when considering insolvency risk, does not allow for an explicit solution, but numerical algorithms can be developed for seeking such solutions (see Appendix 1 for discussion of finite differentiation, simulation and optimization). The results of these numerical calculations are shown below.

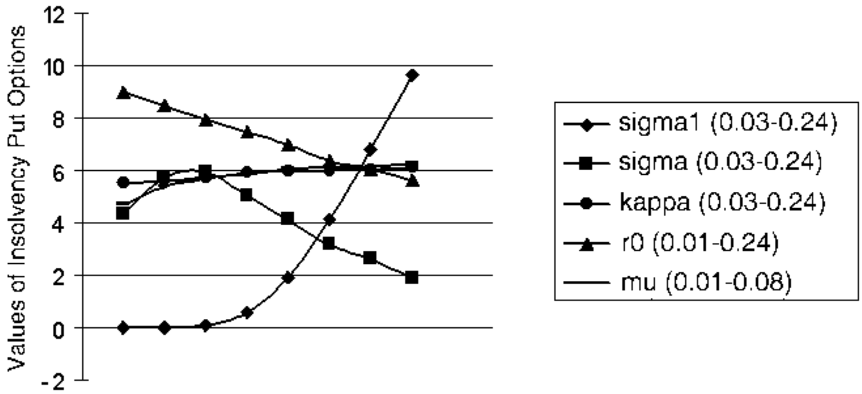
Table 3, along with Figures 1 to 4, displays the pattern of optimal expected net present values, $ENPV(n^*)$, optimal prices $PP(n^*,b(n^*), \pi^1(x,Y))$, and values of the insolvency put option $b(n^*)$, with changes in parameters of the speed of adjustment in the mean reverting prices (κ) initial value of short run interest rate (r_0) volatility of accumulated investment (σ_1), volatility of interest rates (σ), long run equilibrium interest level (μ), and age of the insured (χ).

Figure 1 shows that the value of the insolvency put option is negatively related to the initial value of the short run interest rate (r_0) and the volatility of interest rates (σ, except when σ takes on values less than .09), and positively related to the long run equilibrium interest level (μ, consistent with Browne, Carson, and Hoyt, 1999), the speed of adjustment in the mean reverting price (κ), and the volatility of accumulated investment (σ_1, consistent with Babbel, Jeremy, and Merrill, 2002).

Table 3. Optimal Values of Expected Net Present Value, *ENPV(n*)*; Number of Policies, *n**; Price of Policies *PP(n*,b(n*), π¹(x,Y))*; and Values of Insolvency Put Option *b(n*)*

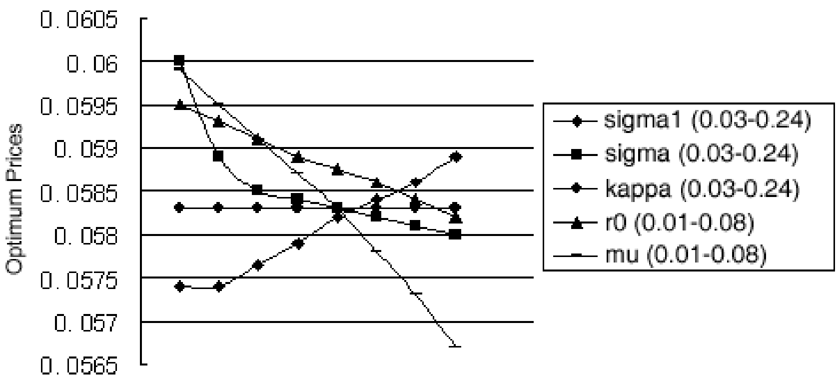
	$x = 49$, $\kappa = 0.24$, $r_0 = 0.07$, $\mu = 0.05$, $\sigma = 0.1$							
σ_1	0.03	0.06	0.09	0.12	0.15	0.18	0.21	0.24
$MaxENVP(n^*)$	212.489	212.489	212.844	211.3196	206.477	198.791	189.248	179.045
n^*	10308	10308	10168	9960	9546	8992	8348	7668
$PP(n^*, b(n^*), \pi^1(x,Y))$	0.0574	0.0574	0.0577	0.0579	0.0582	0.0584	0.0586	0.0589
$b(n^*)$	0	0	0.0863	0.05743	1.9126	4.1299	6.8007	9.6183
	$x = 49$, $\kappa = 0.24$, $r_0 = 0.07$, $\mu = 0.05$, $\sigma = 0.2$							
σ_1	0.03	0.06	0.09	0.12	0.15	0.18	0.21	0.24
$MaxENVP(n^*)$	194.149	192.266	192.578	196.099	202.130	209.771	215.853	222.511
n^*	8146	8422	8620	8816	8964	9266	9348	9584
$PP(n^*, b(n^*), \pi^1(x,Y))$	0.0600	0.0589	0.0585	0.0584	0.0583	0.0582	0.0581	0.0580
$b(n^*)$	4.3204	5.7134	5.9165	5.0325	4.0970	3.1615	2.6064	1.8980
	$x = 49$, $\sigma_1 = 0.2$, $r_0 = 0.07$, $\mu = 0.05$, $\sigma = 0.1$							
κ	0.03	0.06	0.09	0.12	0.15	0.18	0.21	0.24
$MaxENVP(n^*)$	203.133	202.218	199.800	196.876	195.383	194.687	192.870	192.458
n^*	8742	8742	8748	8722	8686	8656	8644	8632
$PP(n^*, b(n^*), \pi^1(x,Y))$	0.0583	0.0583	0.0583	0.0583	0.0583	0.0583	0.0583	0.0583
$b(n^*)$	5.5141	5.6077	5.7292	5.8934	5.9505	5.9703	6.0043	6.1095
	$x = 49$, $\kappa = 0.24$, $\sigma_1 = 0.2$, $\mu = 0.05$, $\sigma = 0.1$							
r_0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08
$MaxENVP(n^*)$	167.359	172.241	176.162	180.609	184.393	188.829	192.458	196.480
n^*	7550	7780	7964	8110	8302	8466	8632	8816
$PP(n^*, b(n^*), \pi^1(x,Y))$	0.0595	0.0593	0.0591	0.0590	0.0588	0.0586	0.0584	0.0582
$b(n^*)$	8.9675	8.4508	7.9340	7.4546	6.9751	6.3606	6.0382	5.6151
	$x = 49$, $\kappa = 0.24$, $r_0 = 0.07$, $\sigma_1 = 0.2$, $\sigma = 0.1$							
μ	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08
$MaxENVP(n^*)$	169.720	174.428	179.522	185.656	192.458	199.830	206.734	214.247
n^*	8276	8324	8462	8558	8632	8892	9124	9392
$PP(n^*, b(n^*), \pi^1(x,Y))$	0.0599	0.0595	0.0591	0.0587	0.0582	0.0578	0.0573	0.0567
$b(n^*)$	4.6834	5.3654	5.6857	6.0059	6.0382	6.0792	6.1202	6.2472
	$\sigma_1 = 0.2$, $\kappa = 0.24$, $r_0 = 0.07$, $\mu = 0.05$, $\sigma = 0.1$							
x	39	41	43	45	47	49	51	53
$MaxENVP(n^*)$	443.917	403.402	358.665	309.256	254.751	192.458	130.412	74.198
n^*	14644	14068	12900	11924	10644	8632	6523	4684
$PP(n^*, b(n^*), \pi^1(x,Y))$	0.0470	0.0483	0.0507	0.0526	0.0549	0.0584	0.0624	0.0653
$b(n^*)$	0.1734	0.4924	1.0036	2.819	3.6308	6.0382	8.8757	11.7133

Other parameters: $AA = 0.075$, $B = 2 \times 10^{-6}$, $G = 0.0004$, $F = 0.10$, $p = 0.2$, $Y = 5$, $\rho_{1,2} = 0.5$, $dt = 1$, mortality data (1980 CSO).



Assumptions for Figure 1 are: $AA = 0.075$, $B = 2\ 10^{-6}$, $G = 0.0004$, $F = 0.1$, $p = 0.2$, $x = 49$, $Y = 5$, $dt = 1$, $\rho_{1,2} = 0.5$, and $q_{49} = 0.00612$, $q_{50} = 0.00663$, $q_{51} = 0.00720$, $q_{52} = 0.00784$, $q_{53} = .00857$

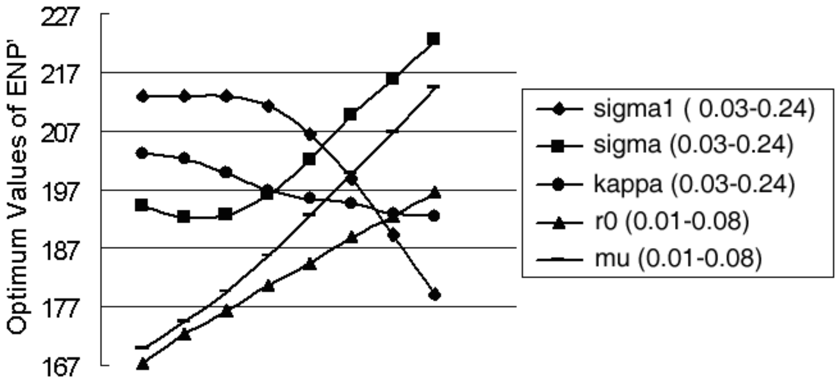
Fig. 1.[3] Values of Insolvency Put Option *b(n*)* versus Parameters of κ, r_0, μ, σ, σ_1



Assumptions for Figure 2 are: $AA = 0.075$, $B = 2 \times 10^{-6}$, $G = 0.0004$, $F = 0.1$, $p = 0.2$, $x = 49$, $Y = 5$, $dt = 1$, $\rho_{1,2} = 0.5$, and $q_{49} = 0.00612$, $q_{50} = 0.00663$, $q_{51} = 0.00720$, $q_{52} = 0.00784$, $q_{53} = .00857$

Fig. 2. Optimal Prices $PP(n^*, b(n^*), \pi^1(x,Y))$ versus Parameters of κ, r_0, μ, σ, σ_1

From Figure 2 we observe that optimal price is most sensitive and negatively related to the long run equilibrium interest level (μ), which ranges from 0.0599 (when $\mu = 0.01$) to 0.0567 (when $\mu = 0.08$). Optimal price also is negatively related to the volatility of interest rates (σ) and the initial value of short run interest rate (r_0). Optimal price is positively related to the volatility of accumulated investment (σ_1). Findings also suggest that



Assumptions for Figure 3 are: AA = 0.075, $B = 2 \times 10^{-6}$, G = 0.0004, F = 0.1, p = 0.2, x = 49, Y = 5, dt = 1, $\rho_{1,2}$ = 0.5, and q_{49} = 0.00612, q_{50} = 0.00663, q_{51} = 0.00720, q_{52} = 0.00784, q_{53} = .00857

Fig. 3. Optimal Expected Net Present Values *ENPV(n*)* versus Parameters of κ, r_0, μ, σ, σ_1

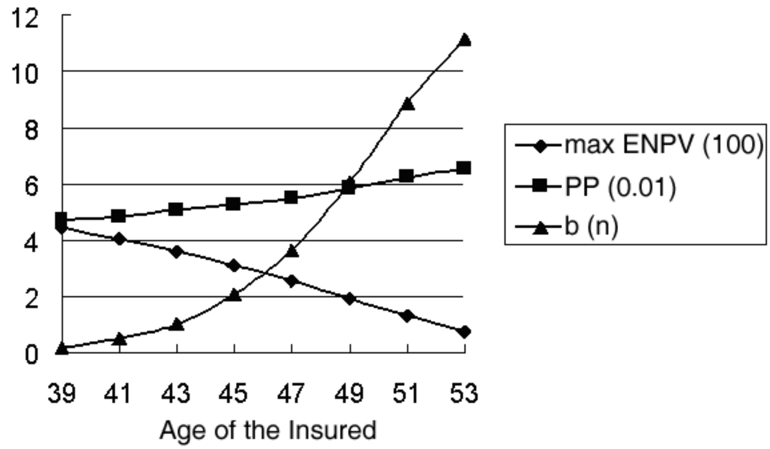
optimal prices are insensitive to the change in the speed of adjustment in the mean reverting prices (κ).

From Figure 3 we observe that optimal net present value is negatively related to the speed of adjustment in the mean reverting prices (κ) and volatility of accumulated investment (σ_1), and positively related to the volatility of interest rates (σ) (when $\sigma \geq 0.06$), long run equilibrium interest level (μ), and the initial value of the short run interest rate (r_0).

Figure 4 indicates that the age of the insured has a positive influence on optimal prices and values of insolvency put option and has a negative influence on optimal expected net present values. The value of insolvency put option is highly sensitive to the change of age of the insured, and the sensitivity rises with the increase of age of the insured (see Appendix 2 for the proof of continuity of Figure 1 to Figure 4).

Table 4 lists the numerical results to demonstrate the influence of values of the insolvency put option on the benefits of the insurer and the insured. We examine the volatility of accumulated investment (σ_1), which is most sensitive to the change in values of put option, and set it at three different levels (0.03, 0.5, 0.8) to see the corresponding changes in the values of the insolvency put option, expected net present values, price, and liability.

From Table 4 we see that when the volatility of the accumulated investments, σ_1, increases, the value of the insolvency put option, $b(n^*)$, also increases. At the same time, both the expected net present value and the value of liability decrease. Looking further, as the values of liability for



Assumptions for Figure 4 are: $AA = 0.075$, $B = 2 \times 10^{-6}$, $G = 0.0004$, $F = 0.1$, $p = 0.2$, $\mu = 0.05$, $\sigma = 0.1$, $\sigma_1 = 0.2$, $x = 39–53$, $Y = 5$, $r_0 = 0.07$, $\kappa = 0.24$, $dt = 1$, and mortality data are from the 1980 CSO Table.

Fig. 4. Optimal Expected Net Present Values ($ENPV(n^*)$), Prices $PP(n^*,b(n^*), \pi^1(x,Y))$, and Values of Insolvency Put Option($b(n^*)$) versus Age of the Insured (x)

the insurer (and insured) decrease, the insolvency put option moves from being far out of the money to being much in the money.

Table 4. Relationship of Values of Insolvency Put Option on Benefits of the Insurer and the Insured

σ_1	Case 1 (0.03)	Case 2 (0.5)	Case 3 (0.8)
$MaxENPV(n^*)$	212.4891	111.9641	91.3571
n^*	10308	4084	2584
$PP(n^*, b(n^*), \pi^1, (x, Y))$	0.0574	0.0552	0.0496
$\pi^1, (x, Y)$	0.0307	0.0307	0.0307
$b(n^*)$	0	36.6580	58.1364
The values of liability	379.7467	113.7466	37.0582
The values of liability for each life contract	0.03684	0.02785	0.01434

Assumptions for Table 4 are: $AA= 0.075$, $B = 2 \times 10^{-6}$, $G = 0.0004$, $F = 0.1$, $p = 0.2$, $x = 49$, $Y = 5$, $dt = 1$, $\kappa = 0.24$, $r_0 = 0.07$, $\mu = 0.05$, $\sigma = 0.1$, $\rho_{1,2} = 0.5$, and $q_{49} = 0.00612$, $q_{50} = 0.00663$, $q_{51} = 0.00720$, $q_{52} = 0.00784$, $q_{53} = .00857$

CONCLUSIONS

The foregoing analysis combined economic, financial, and actuarial approaches in the pricing of single premium and level premium term life insurance. Stochastic control models were developed to maximize the expected net present value of insurer profit considering supply and demand, and risks associated with investments, mortality, interest, and insolvency. Optimal prices were obtained by solving objective functions with optimization techniques and Monte Carlo simulation. The Cox, Ingersoll, Ross (1985) financial valuation model was used in order to calculate expected claim payment and values of insolvency put options. The analysis of the effects of interest rates and other parameters indicates that optimal prices generally are most sensitive to changes in the long run equilibrium interest rate (μ). In addition, the age of the insured (x), volatility of interest rate (σ), and volatility of accumulated investment (σ_1) have strong influence on the value of the insolvency option, especially when they take on larger values. Future research combining economic and actuarial aspects could focus on the pricing of life insurance products such as those with minimum guaranteed returns and annuities, as well as investigating the feasibility of applying these methods in practice. Finally, when determining prices of insurance products, many factors must be considered, including overall business strategy, the dynamics of the marketplace, moral hazard, and adverse selection resulting from information asymmetries.

APPENDIX 1

1. The calculation of π^1

Let the time horizon Y be divided into time intervals s of constant length, then,

$$\pi^1 \approx \sum_{i=1}^{s} A(t_i)e^{B(t_i)r_t} \Delta f_x(t_i) = \sum_{i=1}^{s} A(t_i)e^{B(t_i)r_t} {}_{\Delta t}q_{x+t_{i-1}} , \qquad (19)$$

where ${}_{\Delta t}q_{x+t_{i-1}}$ indicates the probability that the insured will die between ages $x+t_{i-1}$ and $x+t_i$ given that he is alive at age x, and $\Delta t = t_i - t_{i-1}$.

$$r_{t_i} - r_{t_{i-\Delta t}} = \kappa(\mu - r_{t_{i-\Delta t}})\Delta t + \sigma\sqrt{r_{t_{i-\Delta t}}}\varepsilon_1\sqrt{\Delta t} \quad (20)$$

$$r_{t_i} = r_{t_{i-\Delta t}} + \kappa(\mu - r_{t_{i-\Delta t}})\Delta t + \sigma\sqrt{r_{t_{i-\Delta t}}}\varepsilon_1\sqrt{\Delta t},\ i = 1, 2, \ldots s\,,$$

$$D_{t_i} = D_{t_{i-\Delta t}} + r_{t_{i-\Delta t}} D_{t_{i-\Delta t}}\Delta t + \sigma_1 D_{t_{i-\Delta t}}\varepsilon_2\sqrt{\Delta t},$$

$$= D_0\left(\sum_{j=0}^{i-1}(1 + r_{t_i}\Delta t + \sigma_1\varepsilon\sqrt{\Delta t})\right),\ i = 1, 2, \ldots s \quad (21)$$

$$\text{where } \varepsilon_1 = \phi_1,\ \varepsilon_2 = \rho_{1,2}\phi_1 + \sqrt{1 - \rho_{1,2}\phi_2},$$

and ϕ_1, ϕ_2 have the standard and independent normal distributions.

Do simulation of M runs and generate at least $2 \times s$ independent random numbers of standard normal distribution for each simulation run. They are then substituted into the discrete equations (20) and (21).

By combining equations of (20) with (19) repeatedly for M runs to calculate the values of $r_{t_i}^{(1)} \ldots\ r_{t_i}^{(M)},\ i = 1, 2, \ldots\ s; \pi^{1^{(1)}} \ldots\ \pi^{1^{(M)}}$ and letting

$$\pi^1 = \frac{\pi^{1(1)} + \ldots + \pi^{1(M)}}{M}, \quad (22)$$

we can get the values of π^1.

2. The calculation of *b(n)*

$$b(n) \approx E^Q\left[\sum_{i=1}^{s} A(t_i)e^{-B(t_i)r_t}\Delta B_{t_i}\right]$$

$$= E^Q\left[\sum_{i=1}^{s} A(t_i)e^{-B(t_i)r_t}[\max(X_{t_i} - D_{t_i}),0 - \max((X_{t_{i-1}} - D_{t_{i-1}}),0)]\right] \quad (23)$$

When $D_0 = (AA - Bn - Gb(n) + F\pi^1(x, Y))n$,

$$D_{t_i} = [AA - Bn - Gb(n) + F\pi^1(x, Y)]n\sum_{j=0}^{i-1}(1 + r_{t_i}\Delta t + \sigma_1\varepsilon\sqrt{\Delta t}).$$

Equation (23) becomes

$$b(n) - E^Q \left\{ \sum_{i=1}^{s} A(t_i) e^{-B(t_i)r} [\max {}_{t_j}q_x + n(1+p) - \right.$$

$$(AA - Bn - Gb(n) + F\pi^1(x, Y))n \sum_{j=0}^{i-1} (1 + r_{t_j}\Delta t + \sigma_1 \varepsilon \sqrt{\Delta t})),0] \tag{24}$$

$$-\max\Big[{}_{t_{j-1}}q_x n(1+p) - (AA - Bn - Gb(n) + F\pi^1(x, Y))$$

$$\left. n \sum_{j=0}^{i-1} (1 + r_{t_{j-1}}\Delta t + \sigma_1 \varepsilon \sqrt{\Delta t})),0]] \right\} = 0$$

By iterative approximation calculation, we find that the iterative values of *b(n)* are convergent. Therefore, *b(n)* has a unique solution. Table 5 lists iterative (10 times) results setting $AA = 0.075$, $B = 2 \times 10^{-6}$, $G = 0.0004$, $F = 0.1$, $p = 0.2$, $x = 49$, $Y = 5$, $dt = 1$, $\kappa = 0.24$, $r_0 = 0.07$, $\mu = 0.05$, $\sigma = 0.1$, $\sigma_1 = 0.2$, $\rho_{1,2} = 0.5$. Results are based on the same 1,000,000 simulation runs.

Table 5. Numerical Calculation Results of $b(n)$

$b(n)^{(0)}$	$b(n)^{(1)}$	$b(n)^{(2)}$	$b(n)^{(3)}$	$b(n)^{(4)}$	$b(n)^{(5)}$
0	6.3239	6.1137	6.0072	6.0027	6.0128
$b(n)^{(6)}$	$b(n)^{(7)}$	$b(n)^{(8)}$	$b(n)^{(9)}$	$b(n)^{(10)}$	
6.0701	6.0391	6.0189	6.0167	6.0382	

3. The optimization

The process of optimization is described below:

When $PP(n, b(n), \pi^1(x, Y)) = AA - Bn - Gb(n) + F\pi^1(x, Y)$ and $b(n) = 0$, the constrain condition for objective function *Max ENPV(n)* becomes $AA - Bn + F\pi^1(x, Y) > 0$, or $n < \dfrac{AA + F\pi^1(x, Y)}{B}$. While it is easily known from equation (6) that $b(n) \geq 0$ (the integrand and dH_t are larger than or equal to 0 and $Y > 0$), and the upper boundary limit of n for $MaxENPV(n)$ is less than or equal to $\frac{AA + F\pi^1(x, Y)}{B}$, a search procedure is employed to locate optimum solution n^*, which satisfies constraint $AA - Bn + Gb(n) + F\pi^1(x, Y) > 0$ in the interval of $\left(0, \frac{AA + F\pi^1(x, Y)}{B}\right)$ for n.

APPENDIX 2

The proof of continuity of equation (6):
Equation (6) can be written as:

$$b(n) = E^Q\left[\int_0^Y A(t)e^{-B(t)r} d\max({}_tq_x n(1+p) - D_t, 0)\right] \tag{25}$$

$$= \int_0^Y A(t)e^{-B(t)r} dE^Q[\max({}_tq_x n(1+p) - D_t, 0)].$$

From equation (25), we can know that the first term of the integrand of $b(n)$— $A(t)e^{-B(t)r}$—is a continuous function,

$$\text{where } A(t) = \left[\frac{2\gamma e^{(\kappa+\gamma)t/2}}{g(t)}\right]^{2\kappa\mu/\sigma^2}$$

$$B(t) = \frac{2(e^{\gamma t} - 1)}{g(t)}$$

$$g(t) = 2\gamma + (\kappa + \gamma)(e^{\gamma t} - 1)$$

$$\gamma = \sqrt{\kappa^2 + 2\sigma^2}.$$

Let $R = {}_tq_x n(1+p) - D_t$, then

$$dE^Q[\max({}_tq_x n(1+p) - D_t, 0)] = \frac{dE^Q[\max(R,0)]}{dt}dt.$$

$$\text{when } R \geq 0, dE^Q \max(R,0) = \frac{d}{dt}E^Q\left[{}_tq_x n(1+p)dt - \frac{dE^Q(D_t)}{dt}dt\right]$$

$$= f_x(t)n(1+p)dt + E^Q\left(rD_t dt - \sigma_1 D_t \frac{dw^1}{dt}dt\right)$$

$$\leq f_x(t)n(1+p)dt + E^Q(rD_t dt) - \sqrt{E^Q(\sigma_1 D_t)^2}\sqrt{\frac{d}{dt}E^Q(w^1)^2 dt}.$$

Since the variance of standard Wiener process $D^Q(w^1) = t$, the expectation of standard Wiener process $E^Q(w^1) = 0$ and

$$E^Q(w^1)^2 = D^Q(w^1) - [E^Q(w^1)]^2 = t,$$

$$dE^Q \max(R,0) \leq f_x(t)n(1+p)dt + E^Q(rD_t)dt - \sqrt{E^Q(\sigma_1 D_t)^2}\sqrt{\frac{d}{dt}E^Q(w^1)^2 dt}$$

$$= f_x(t)n(1+p)dt + E^Q(rD_t)dt - \sqrt{E^Q \sigma_1 D_t^2}\sqrt{dt}.$$

When R < 0, $dE^Q \max(R,0) = 0$.

Therefore $dE^Q \max(R,0)$ is differentiable and continuous. And since $A(t)e^{-B(t)r}$ is also continuous, $b(n)$ is a continuous function.

While the price of the life insurance contract and the expected net present value are functions of $b(n)$ and $\pi^1(x, Y)$, and $\pi^1(x, Y)$ is a continuous function, $ENPV(n)$ and $PP(n, b(n), \pi^1, (x, Y))$ also are continuous functions.

Therefore, we can derive their graphs from a selected discrete set of points at which calculations for Figure 1 through Figure 4 were performed.

ENDNOTES

[1] In the Vasicek (1977) and Langetieg (1980) models (see also Yao, 1999), interest rates are normally distributed and there is a positive probability of negative interest rates (which implies arbitrage opportunities). These two models are not used in our analysis.

[2] The put option is often called the insolvency put because it is exercised only if the firm is insolvent (Cummins, 1991).

[3] For Figure 1 through Figure 4, the functions are continuous, and numerical experiments suggest monotonicity. Figures are shown to illustrate the results of the methods developed here, and are based on assumptions believed to be reasonable.

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