ELEMENTARY LINEAR ALGEBRA

A MATRIX APPROACH, 2e

LAWRENCE E. SPENCE

ARNOLD J. INSEL

STEPHEN H. FRIEDBERG

Illinois State University

PEARSON
Prentice Hall

Elementary Linear Algebra, Second Edition,
by Spence, Insel, and Friedberg. ISBN 0-13-187141-2.

APPENDIX: Mathematical Proof

There are many mathematical statements whose truth is not obvious. For example, the French mathematician Pierre de Fermat (1600–1665) asserted:

There exist no positive integers x, y, z, and n, with $n > 2$, such that $x^n + y^n = z^n$.

In spite of much effort to prove Fermat's statement, it remained unproved until 1993.

A sophomore-level linear algebra course is often the first mathematics course in which there is a significant emphasis on theory and proof. In this appendix, we briefly discuss some ideas that are helpful in such a course.

A *definition* is a name for an object or a property; so a definition does not require proof. Definitions are extremely important because they provide the vocabulary that permits effective communication. For example, on page 103 a matrix is defined to be *symmetric* if it is equal to its transpose. This definition tells us that a matrix A is symmetric precisely when it satisfies the condition that $A^T = A$. So, if we wish to prove that a matrix is symmetric, we must show that it is equal to its transpose, and nothing more.

On the other hand, a *theorem* is a statement that asserts something is true. A theorem requires proof. Other words that can be used in place of *theorem* are *principle*, *fact*, or *result*; such statements also require proofs. Sometimes the word *proposition* is used instead of *theorem* for the statement of a minor result. Also, the word *lemma* is used for a statement that is of little significance of its own but which is useful in proving another result, and the word *corollary* is used for a result that follows directly from another.

Statements

A sentence that is either true or false, but not both, is called a *statement*. Two or more statements are said to be *equivalent* (or *logically equivalent*) if, in every circumstance, they are all true or all false. Thus we can show that a statement is true by proving instead that an equivalent statement is true, and we can replace any statement in a proof by a logically equivalent statement.

The statement

"P if and only if Q"

means that P and Q are logically equivalent.

Mathematical theorems are frequently of the form

"if P, then Q," (1)

where P and Q are statements. Another way of writing this is

"P implies Q,"

and so such statements are called *implications*. In either of these forms, P is called the *hypothesis* and Q is called the *conclusion*.

Elementary Linear Algebra, Second Edition,
by Spence, Insel, and Friedberg. ISBN 0-13-187141-2.

Example 1 Theorem 1.1(a) states:

If A and B are $m \times n$ matrices, then $A + B = B + A$.

Identify the hypothesis and conclusion of this implication.

Solution The hypothesis is "A and B are $m \times n$ matrices," and the conclusion is "$A + B = B + A$."

Be aware, however, that the flexibility of the English language permits us to write an implication without the *if ... then ...* structure. For example, Theorem 1.1(a) could also be expressed as follows:

For all $m \times n$ matrices A and B, $A + B = B + A$.

In Bloomington, Illinois, the Director of Public Service is permitted to call a temporary parking ban to assist with snow removal. The ordinance states

If a parking ban is in effect, then cars may not be parked on designated streets.

This statement is of the form "if P, then Q," where

P is the statement "A parking ban is in effect"

and

Q is the statement "Cars may not be parked on designated streets."

For this ordinance to be violated, a parking ban must be in effect and a car must be parked on a designated street. Thus the ordinance is violated only when P is true and Q is false. In a similar way, logicians and mathematicians regard the statement "if P, then Q" to be true unless P is true and Q is false.

> The implication "if P, then Q" is true except when P is true and Q is false.

Two variations of statement (1) occur often. They are

"if Q, then P" (the *converse* of statement (1))

and

"if not Q, then not P." (the *contrapositive* of statement (1)).

Example 2 Give the converse and contrapositive of the statement

If $x = 0$, then $x^2 = x$.

Solution The hypothesis of the given statement is "$x = 0$," and the conclusion is "$x^2 = x$." Interchanging the hypothesis and conclusion gives the converse, which is

$$\text{If } x^2 = x, \text{ then } x = 0.$$

Interchanging the hypothesis and conclusion of the original statement and negating each of them gives the contrapositive, which is

$$\text{If } x^2 \neq x, \text{ then } x \neq 0.$$

A statement and its converse may be both true, both false, or one may be true and the other false. In Example 2, the original statement is true and its converse is false. On the other hand, the statement

$$\text{If } x = 0, \text{ then } x^2 = 0$$

and its converse are both true, and the statement

$$\text{If } x = 1, \text{ then } x^2 = 0$$

and its converse are both false.

Although either a statement or its converse may be true and the other false, a statement and its *contrapositive* always have the same truth value.

A statement and its contrapositive are logically equivalent.

Statements Involving Quantifiers

Many mathematical statements involve variables that are qualified by the words "for all" (or "for every" or "for each") and "there exists." Suppose we working with the integers, that is, the whole numbers

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots.$$

Consider the following statements.

(A) For every integer $n, 2n + 7$ is odd.

(B) For every integer $n, n^3 \geq n$.

(C) There exists an integer n such that $(n^2 + 1)/3$ is an integer.

(D) There exists an integer n such that $n^3 + 34 = n^2 + n + 1$.

Elementary Linear Algebra, Second Edition,
by Spence, Insel, and Friedberg. ISBN 0-13-187141-2.

Statement (A) amounts to infinitely many assertions, namely, that

$$\begin{aligned} 2 \cdot 0 + 7 &= \ \ 7 \text{ is odd,} \\ 2 \cdot 1 + 7 &= \ \ 9 \text{ is odd,} \\ 2 \cdot (-1) + 7 &= \ \ 5 \text{ is odd,} \\ 2 \cdot 2 + 7 &= 11 \text{ is odd,} \\ &\vdots \end{aligned}$$

The fact that 7, 9, 5, and 11 are all odd does not prove statement (A), although it might lead us to suspect its truth.

Likewise statement (B) asserts that

$$\begin{aligned} 0^3 &\geq 0, \\ 1^3 &\geq 1, \\ (-1)^3 &\geq -1, \\ 2^3 &\geq 2, \\ &\vdots \end{aligned}$$

The four preceding statements are all true, but again this is not sufficient to prove statement (B). In fact, taking $n = -2$ leads to the false statement

$$(-2)^3 \geq -2, \qquad \text{that is,} \qquad -8 \geq -2.$$

Thus statement (B) is false because it is not true for *every* integer n. This illustrates the following fact.

Only a single counterexample is needed to show that a "for all" statement is false.

The truth of a statement of existence, such as (C) or (D), is usually easier to confirm than that of a "for all" statement. For example, since

$$(-3)^3 + 34 = 7 = (-3)^2 + (-3) + 1,$$

statement (D) is true. That is, statement (D) is true because -3 is an integer, and $n^3 + 34 = n^2 + n + 1$ for $n = -3$. This illustrates the most common way of confirming the truth of a "there exists" statement, namely, by actually exhibiting an object of the type that is claimed to exist.

Elementary Linear Algebra, Second Edition,
by Spence, Insel, and Friedberg. ISBN 0-13-187141-2.

Showing that a “there exists” statement is false is more difficult, however. For example, none of

$$\frac{(0^2+1)}{3}, \quad \frac{(1^2+1)}{3}, \quad \frac{[(-1)^2+1]}{3}, \quad \frac{(2^2+1)}{3}, \quad \text{and} \quad \frac{[(-2)^2+1]}{3}$$

are integers. Yet these five examples do not show that for *no* integer n is $(n^2+1)/3$ an integer, and so they do not in themselves prove that statement (C) is false.

The negation of a “for all” statement is a “there exists” statement, and the negation of a “there exists” statement is a “for all” statement. For example, the following are the negations of the previous statements (A) through (D).

(A′) There exists an integer n such that $2n+7$ is not odd.

(B′) There exists an integer n such that $n^3 < n$.

(C′) For every integer n, $(n^2+1)/3$ is not an integer.

(D′) For every integer n, $n^3+34 \neq n^2+n+1$.

Example 3 Write the negation of each of the following statements.

(a) There exists a nonzero real number x such that $x^2 = 2x$.

(b) For every nonzero real number x, $x^2 \geq x$.

Solution (a) To form the negation, replace *there exists* by *for every* and negate the statement “$x^2 = 2x$.” The resulting statement is

For every nonzero real number x, $x^2 \neq 2x$.

(b) To form the negation, replace *for every* by *there exists* and negate the statement “$x^2 \geq x$.” The resulting statement is

There exists a nonzero real number x such that $x^2 < x$.

Direct Proofs

As we previously noted, mathematical theorems often have the form of an implication “if P, then Q.” To prove such a statement, we start with its hypothesis and, using valid logic and results that are known or assumed to be true, deduce its conclusion. Because a false statement logically implies any other statement, no matter whether the latter is true or false, we have the following inviolable rule.

A proof must begin with a statement that is known or assumed to be true.

In particular, it is *never* permissible to begin a proof with the result that is being proved.

The simplest method for proving the implication “if P, then Q” is a *direct proof*, where we assume that P is true and deduce that Q is true.

Elementary Linear Algebra, Second Edition,
by Spence, Insel, and Friedberg. ISBN 0-13-187141-2.

We now give a few examples of direct proofs. These involve only a few familiar properties of the integers. Recall that an integer n is called *even* if $n = 2k$ for some integer k, and it is called *odd* if $n = 2k + 1$ for some integer k. We assume it is known that every integer is either even or odd, but not both, and that sums and products of integers are integers.

PROPOSITION 1 For any integer n, if n is odd, then $3n$ is odd.

To prove Proposition 1 by a direct proof, we must assume that n is an odd integer and deduce that $3n$ is an odd integer. For n to be an odd integer means that $n = 2k + 1$ for some integer k. We must show that $3n$ has a similar form, that is, $3n = 2j + 1$ for some integer j. In this case, the key idea in the proof is to discover the value of j. A simple calculation enables us to do this:

$$3n = 3(2k + 1) = 6k + 3.$$

For $3n$ to equal $2j + 1$, we must have

$$\begin{aligned} 3n &= 2j + 1 \\ 6k + 3 &= 2j + 1 \\ 6k + 2 &= 2j \\ 2(3k + 1) &= 2j \\ 3k + 1 &= j. \end{aligned}$$

Note that $3k + 1$ is a sum and product of integers, and so is an integer. If, therefore, we take j to be the integer $3k + 1$, then $3n = 2j + 1$.

When formally writing the proof, we don't need to show all the details that led us to discover the value of j; we need only show that the value $j = 3k + 1$ has the desired property. So a formal proof might look something like this.

PROOF Let n be an odd integer. Then $n = 2k + 1$ for some integer k. Therefore

$$3n = 3(2k + 1) = 6k + 3 = (6k + 2) + 1 = 2(3k + 1) + 1.$$

Because $j = 3k + 1$ is a sum and product of integers, it is an integer. Thus $3n$ equals 2 times the integer j plus 1, and so $3n$ is odd. ■

The converse of Proposition 1 is also true.

PROPOSITION 2 For any integer n, if $3n$ is odd, then n is odd.

A direct proof of Proposition 2 would assume that $3n = 2k + 1$ for some integer k, and then use an argument like that in the proof of Proposition 1 to show that n has a similar form. This method does not work here, primarily because to change $3n$ into n, we must *divide* both sides of an equation by 3, unlike the proof of Proposition 1, where we *multiplied* both sides of an equation by 3. (Dividing by 3 is not as algebraically simple as multiplying by 3. For instance, if we divide an integer by 3, the quotient may not be an integer.) Instead, we can prove Proposition 2 by verifying its contrapositive.

Elementary Linear Algebra, Second Edition,
by Spence, Insel, and Friedberg. ISBN 0-13-187141-2.

PROOF We prove the contrapositive: For any integer n, if n is not odd, then $3n$ is not odd. [If you are going to use a proof other than a direct proof, it is helpful to tell the reader so at the start.] Because every integer is either even or odd, but not both, we can restate the contrapositive as: For any integer n, if n is even, then $3n$ is even.

Let n be an even integer, so that $n = 2k$ for some integer k. Then

$$3n = 3(2k) = 2(3k).$$

Because $3k$ is a product of integers, it is an integer. Thus $3n$ equals twice the integer $3k$, and so $3n$ is even. ∎

Although the proofs of Propositions 1 and 2 are both short, they illustrate several good practices.

1. It is helpful to begin by indicating to the reader what type of proof we are using (although this is not usually done if we are giving a direct proof of the result as originally stated).

2. Before doing anything else, we state our assumptions.

3. If we introduce a new symbol during the proof (as we did in Proposition 1 with the symbol j), we immediately write the meaning of the new symbol. Without this information, whatever you write won't be understood by someone reading your proof.

4. Whenever possible, we help the reader to understand the reason that something we say is true. Sometimes we do this by explicitly giving a reason, as we do in the proof of Proposition 1 when we say *because* $j = 3k + 1$ *is a sum and product of integers.* Other times, however, we help the reader implicitly by our choice of words. Using the words *let*, *assume*, or *suppose*, for instance, tells the reader that we are making an assumption. By contrast, words such as *then*, *so*, *thus*, *hence*, and *therefore* indicate to the reader that a conclusion is being drawn from an earlier statement. A proof is much easier to understand if such words are used often.

Notice the use of these practices in the proof that follows.

PROPOSITION 3 For any integer n, if n^2 is even, then n is even.

PROOF We prove the contrapositive, namely, that if n is not even, then n^2 is not even. Note that if n is not even, then n is odd, and so $n = 2k + 1$ for some integer k. Thus

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Because $2k^2 + 2k$ is a sum and product of integers, it is an integer. Hence n^2 is odd since it is of the form $2j + 1$ for the integer $j = 2k^2 + 2k$. Therefore n^2 is not even. ∎

Elementary Linear Algebra, Second Edition,
by Spence, Insel, and Friedberg. ISBN 0-13-187141-2.

Since the statement of Proposition 1 and its converse (Proposition 2) are both true, we could have combined them into the following result.

PROPOSITION 4 For any integer n, n is odd if and only if $3n$ is odd.

In general, to prove a theorem of the form "P if and only if Q," we need to prove both the implications

"if P, then Q" and "if Q, then P."

Thus, to prove Proposition 4, we would need to combine the proofs of Propositions 1 and 2. Here is one form that such a proof might take.

PROOF Assume first that n is an odd integer. Then $n = 2k + 1$ for some integer k. Therefore

$$3n = 3(2k + 1) = 6k + 3 = (6k + 2) + 1 = 2(3k + 1) + 1.$$

Because $j = 3k + 1$ is a sum and product of integers, it is an integer. Thus $3n$ equals 2 times the integer j plus 1, and so $3n$ is odd.

To prove the converse, we show that for any integer n, if n is even, then $3n$ is even. Let n be an even integer, so that $n = 2k$ for some integer k. Then $3n = 3(2k) = 2(3k)$. Because $3k$ equals a product of integers, it is an integer. Thus $3n$ equals twice the integer $3k$, and so $3n$ is even.

The two preceding paragraphs prove Proposition 4. ∎

Proofs by Contradiction

A different style of proof is a *proof by contradiction*, also called an *indirect proof.* To prove the statement "if P, then Q" by contradiction, we assume both that P is true and Q is false and (using statements that are known or assumed to be true and logic that is valid) derive a *contradiction*, which is a statement that is known to be false. Because our assumptions have led to a statement that is false, it cannot be the case that P is true and Q is false. Thus, if P is true, necessarily Q must also be true. In other words, the statement "if P, then Q" is true.

In a direct proof of the statement "if P, then Q", we assume only that P is true, but in a proof by contradiction, we assume both that P is true and Q is false. Because we are allowed to assume more, there are situations where a proof by contradiction is successful but a direct proof is not. However, a proof by contradiction is often more complicated than a direct proof, and the additional assumption is not often needed. So a direct proof should usually be tried before considering a proof by contradiction.

As an example of a proof by contradiction, we give a classical proof that was known to the Greeks at the time of Pythagoras.

PROPOSITION 5 If $r = \frac{a}{b}$, where a and b are integers and $b \neq 0$, then $r^2 \neq 2$. That is, $\sqrt{2}$ is irrational.

PROOF We give a proof by contradiction. Assume that a and b are integers such that $b \neq 0$,

$$r = \frac{a}{b},$$

and $r^2 \neq 2$ is false, so that $r^2 = 2$. Since we could cancel any common factors of 2 in the integers a and b, we can assume that at least one of a and b is odd.

Elementary Linear Algebra, Second Edition,
by Spence, Insel, and Friedberg. ISBN 0-13-187141-2.

Because $r^2 = 2$, we have

$$2 = r^2 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$$
$$2b^2 = a^2.$$

Thus a^2 is even, and so Proposition 3 tells us that a must also be even. Let $a = 2k$, where k is an integer. From the last equation, we see that

$$2b^2 = a^2 = (2k)^2 = 4k^2,$$
$$b^2 = 2k^2.$$

Thus b^2 is even, and so b is also even, again by Proposition 3.

In the preceding paragraph, we proved that both a and b are even. But this contradicts our assumption that at least one of a and b is odd. This contradiction proves Proposition 5. ∎

Writing Proofs

For a serious student of mathematics, writing proofs probably causes more frustration than any other part of learning the subject. Like anything else of importance, it involves failure as well as success. It also takes time—you will not just wake up one morning and be able to write proofs. A proof may contain something unexpected that even an intelligent and conscientious person might overlook. So you cannot expect to write proofs in a mechanical fashion, the way a competent calculus student might work differentiation problems.

The most important step for beginners is to learn what is a proof, and what is not. The standards are absolute; every detail must be correct. This requires worrying about things that you might have previously ignored. For example, if you divide by a real number, you must show that it is nonzero. Every case is important. However, you do not need to be a genius to write proofs. Not everyone is a Michelangelo, but everyone can learn to apply paint to a canvas.

How do you start when creating a proof? To begin, you must completely understand what you are trying to prove and what is assumed. You must also know the definitions of any technical terms that are involved. Unless you can write down from memory exactly what you are trying to prove, you probably do not understand the problem well enough to solve it.

Don't get locked into a single approach when trying to write a proof. Let your mind wander free; consider a variety of ideas and methods. Could a recent or related theorem in the book be put to use? Can a previous exercise be applied? Of course, not all of your ideas will work, but you have to start somewhere. Writing proofs requires a strange combination of the free conceptual flow of an artist and the rigid standards of an IRS auditor. You need the flexibility of mind to come up with original ideas, and the discipline to follow them through to the last detail.

You should not expect to write down the final version of even the most straightforward proof in only one attempt, any more than you would prepare a term paper and submit it without corrections or revisions. A good proof evolves from your original idea in a sequence of refinements.

When learning to write proofs, it is especially important to study examples of proofs. Your mathematics textbooks and instructors provide models of how a good proof should be written. Yet as helpful as studying correct proofs can be, you will learn to write proofs only by attempting them. No one would expect to learn to swim or play the piano merely from reading books, and the same is true for constructing proofs. Yet just as a baseball player who gets one hit in every three at bats is considered an excellent hitter, no one should expect to bat 1.000 in creating proofs. Mistakes are to be expected, and even the best mathematicians make them occasionally. Eventually, however, success will become more frequent, and you will develop the confidence, inventiveness, and flexibility of mind that enable you to write proofs.

Elementary Linear Algebra, Second Edition,
by Spence, Insel, and Friedberg. ISBN 0-13-187141-2.