Mathematics in Ancient India

1. An Overview

Amartya Kumar Dutta

Amartya Kumar Dutta is an Associate Professor of Mathematics at the Indian Statistical Institute, Kolkata. His research interest is in commutative algebra.

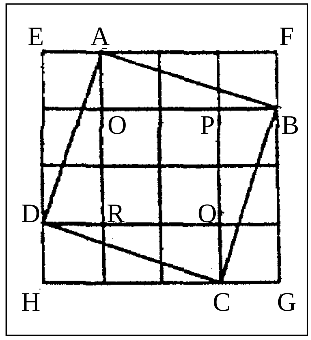
In this series of articles, we intend to have a glimpse of some of the landmarks in ancient Indian mathematics with special emphasis on number theory. This issue features a brief overview of some of the high peaks of mathematics in ancient India. In the next part we shall describe Aryabhata's general solution in integers of the equation $ax - by = c$. In subsequent instalments we shall discuss in some detail two of the major contributions by Indians in number theory. The climax of the Indian achievements in algebra and number theory was their development of the ingenious *chakravala* method for solving, in integers, the equation $x^2 - Dy^2 = 1$, erroneously known as the Pell equation. We shall later describe the partial solution of Brahmagupta and then the complete solution due to Jayadeva and Bhaskaracharya.

Vedic Mathematics: The Sulba Sutras

Mathematics, in its early stages, developed mainly along two broad overlapping traditions: (i) the geometric and (ii) the arithmetical and algebraic. Among the pre-Greek ancient civilizations, it is in India that we see a strong emphasis on both these great streams of mathematics. Other ancient civilizations like the Egyptian and the Babylonian had progressed essentially along the computational tradition. A Seidenberg, an eminent algebraist and historian of mathematics, traced the origin of sophisticated mathematics to the originators of the Rig Vedic rituals ([1, 2]).

Keywords.
Taittiriya Samhita, Sulba-sutras, Chakravala method, Meru-Prastara, Vedic altars, Yukti-bhasa, Madhava series.

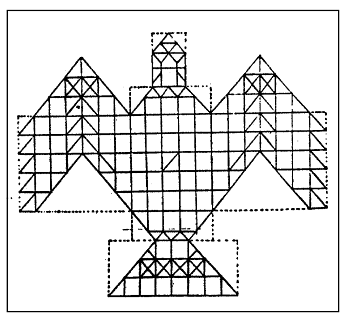
The oldest known mathematics texts in existence are the *Sulba-sutras* of Baudhayana, Apastamba and Katyayana which form part of the literature of the Sutra period of the later Vedic age. The Sulbasutras had been estimated to have been composed around 800 BC (some recent researchers are suggesting earlier dates). But the mathematical knowledge recorded in these *sutras* (aphorisms) are much more ancient; for the Sulba authors emphasise that they were merely stating facts already known to the composers of the Brahmanas and Samhitas of the early Vedic age ([3], [1], [2]).



From the* KĀTYĀYANA *sulba.

The Sulbasutras give a compilation of the results in mathematics that had been used for the designing and constructions of the various elegant Vedic fire-altars right from the dawn of civilization. The altars had rich symbolic significance and had to be constructed with accuracy. The designs of several of these brick-altars are quite involved – for instance, there are constructions depicting a falcon in flight with curved wings, a chariot-wheel complete with spokes or a tortoise with extended head and legs! Constructions of the fire-altars are described in an enormously developed form in the *Satapatha Brahmana* (c. 2000 BC; vide [3]); some of them are mentioned in the earlier *Taittiriya Samhita* (c. 3000 BC; vide [3]); but the sacrificial fire-altars are referred – without explicit construction – in the even earlier Rig Vedic Samhitas, the oldest strata of the extant Vedic literature. The descriptions of the fire-altars from the Taittiriya Samhita onwards are exactly the same as those found in the later Sulbasutras.

Vakrapaksa-syenacit. First layer of construction (after Baudhayana)



Plane geometry stands on two important pillars having applications throughout history: (i) the result popularly known as the 'Pythagoras theorem' and (ii) the properties of similar figures. In the Sulbasutras, we see an explicit statement of the Pythagoras theorem and its applications in various geometric constructions such as construction of a square equal (in area) to the

"How great is the science which revealed itself in the Sulba, and how meagre is my intellect! I have aspired to cross the unconquerable ocean in a mere raft."
B Datta alluding to Kalidasa

sum, or difference, of two given squares, or to a rectangle, or to the sum of n squares. These constructions implicitly involve application of algebraic idenititites such as $(a \pm b)^2 = a^2 + b^2 \pm 2ab$, $a^2 - b^2 = (a + b)(a - b)$, $ab = ((a + b)/2)^2 - ((a - b)/2)^2$ and $na^2 = ((n + 1)/2)^2a^2 - ((n - 1)/2)^2a^2$. They reflect a blending of geometric and subtle algebraic thinking and insight which we associate with Euclid. In fact, the Sulba construction of a square equal in area to a given rectangle is exactly the same as given by Euclid several centuries later ! There are geometric solutions to what are algebraic and number-theoretic problems.

Pythagoras theorem was known in other ancient civilizations like the Babylonian, but the emphasis there was on the numerical and not so much on the proper geometric aspect while in the Sulbasutras one sees depth in both aspects – especially the geometric. This is a subtle point analysed in detail by Seidenberg. From certain diagrams described in the Sulbasutras, several historians and mathematicians like Burk, Hankel, Schopenhauer, Seidenberg and Van der Waerden have concluded that the Sulba authors possessed proofs of geometrical results including the Pythagoras theorem – some of the details are analysed in the pioneering work of Datta ([2]). One of the proofs of the Pythagoras theorem, easily deducible from the Sulba verses, is later described more explicitly by Bhaskara II (1150 AD).

"But the Vedic Hindu, in his great quest of the *Para-vidya, Satyasya Satyam*, made progress in the *Apara-Vidya*, including the various arts and sciences, to a considerable extent, and with a completeness which is unparallelled in antiquity."
B Datta

Apart from the knowledge, skill and ingenuity in geometry and geometric algebra, the Vedic civilization was strong in the computational aspects of mathematics as well – they handled the arithmetic of fractions as well as surds with ease, found good rational approximations to irrational numbers like the square root of 2, and, of course, used several significant results on mensuration.

An amazing feature of all ancient Indian mathematical literature, beginning with the Sulbasutras, is that they

they are composed entirely in verses – an incredible feat! This tradition of composing terse *sutras*, which could be easily memorised, ensured that, inspite of the paucity and perishability of writing materials, some of the core knowledge got orally transmitted to successive generations.

"nor did he [Thibaut] formulate the obvious conclusion, namely, that the Greeks were not the inventors of plane geometry, rather it was the Indians."
A Seidenberg

Post-Vedic Mathematics

During the period 600 BC-300 AD, the Greeks made profound contributions to mathematics – they pioneered the axiomatic approach that is characteristic of modern mathematics, created the magnificent edifice of Euclidean geometry, founded trigonometry, made impressive beginnings in number theory, and brought out the intrinsic beauty, elegance and grandeur of pure mathematics. Based on the solid foundation provided by Euclid, Greek geometry soared further into the higher geometry of conic sections due to Archimedes and Apollonius. Archimedes introduced integration and made several other major contributions in mathematics and physics. But after this brilliant phase of the Greeks, creative mathematics virtually came to a halt in the West till the modern revival.

On the other hand, the Indian contribution, which began from the earliest times, continued vigorously right up to the sixteenth century AD, especially in arithmetic, algebra and trigonometry. In fact, for several centuries after the decline of the Greeks, it was only in India, and to some extent China, that one could find an abundance of creative and original mathematical activity. Indian mathematics used to be held in high esteem by contemporary scholars who were exposed to it. For instance, a manuscript found in a Spanish monastery (976 AD) records ([4],[5]): *"The Indians have an extremely subtle and penetrating intellect, and when it comes to arithmetic, geometry and other such advanced disciplines, other ideas must make way for theirs. The best proof of*

"Anyway, the damage had been done and the Sulvasutras have never taken the position in the history of mathematics that they deserve."
A Seidenberg

"The cord stretched in the diagonal of a rectangle produces both (areas) which the cords forming the longer and shorter sides of an oblong produce separately." (translation from the Sulbasutras)

this is the nine symbols with which they represent each number no matter how large." Similar tribute was paid by the Syrian scholar Severus Sebokht in 662 AD ([5], [6]).

The Decimal Notation and Arithmetic

India gave to the world a priceless gift – the decimal system. This profound anonymous Indian innovation is unsurpassed for sheer brilliance of abstract thought and utility as a practical invention. The decimal notation derives its power mainly from two key strokes of genius: the concept of place-value and the notion of zero as a digit. G B Halsted ([7]) highlighted the power of the place-value of zero with a beautiful imagery: *"The importance of the creation of the zero mark can never be exaggerated. This giving to airy nothing, not merely a local habitation and a name, a picture, a symbol, but helpful power, is the characteristic of the Hindu race whence it sprang. It is like coining the Nirvana into dynamos. No single mathematical creation has been more potent for the general on-go of intelligence and power."*

"A common source for the Pythagorean and Vedic mathematics is to be sought either in the Vedic mathematics or in an older mathematics very much like it. ... What was this older, common source like? I think its mathematics was very much like what we see in the Sulvasutras."
A Seidenberg

The decimal system has a deceptive simplicity as a result of which children all over the world learn it even at a tender age. It has an economy in the number of symbols used as well as the space occupied by a written number, an ability to effortlessly express arbitrarily large numbers and, above all, computational facility. Thus the twelve-digit Roman number (DCCCLXXXVIII) is simply 888 in the decimal system!

Most of the standard results in basic arithmetic are of Indian origin. This includes neat, systematic and straightforward techniques of the fundamental arithmetic operations: addition, subtraction, multiplication, division, taking squares and cubes, and extracting square and cube roots; the rules of operations with fractions and surds; various rules on ratio and proportion like the rule of three; and several commercial and related problems like income and expenditure, profit and loss, simple and compound interest, discount, partnership, computations of the average impurities of gold, speeds and distances, and the mixture and cistern problems similar to those found in modern texts. The Indian methods of performing long multiplications and divisions were introduced in Europe as late as the 14th century AD. We have become so used to the rules of operations with fractions that we tend to overlook the fact that they contain ideas which were unfamiliar to the Egyptians, who were generally proficient in arithmetic, and the Greeks, who had some of the most brilliant minds in the history of mathematics. The rule of three, brought to Europe via the Arabs, was very highly regarded by merchants during and after the renaissance. It came to be known as the Golden Rule for its great popularity and utility in commercial computations – much space used to be devoted to this rule by the early European writers on arithmetic.

"... the basic point is that the dominant aspect of Old Babylonian mathematics is its computational character ... The Sulvasutras know both aspects (geometric and computational) and so does the Satapatha Brahmana."
A Seidenberg

The excellence and skill attained by the Indians in the foundations of arithmetic was primarily due to the advantage of the early discovery of the decimal notation – the key to all principal ideas in modern arithmetic. For instance, the modern methods for extracting square and cube roots, described by Aryabhata in the 5th century AD, cleverly use the ideas of place value and zero and the algebraic expansions of $(a+b)^2$ and $(a+b)^3$. These methods were introduced in Europe only in the 16th century AD. Apart from the exact methods, Indians also invented several ingenious methods for determination of approximate square roots of non-square numbers, some of which we shall mention in a subsequent issue.

"The striking thing here is that we have a proof. One will look in vain for such things in Old-Babylonia. The Old-Babylonians, or their predecessors, must have had proofs of their formulae, but one does not find them in Old-Babylonia."
A Seidenberg
referring to a verse in the Apastamba Sulbasutra on an isoceles trapezoid)

Due to the gaps in our knowledge about the early phase of post-Vedic Indian mathematics, the precise details regarding the origin of decimal notation is not known. The concept of zero existed by the time of Pingala (dated 200 BC). The idea of place-value had been implicit in

"The diagonal of a rectangle produces both (areas) which its length and breadth produce separately." (original verse from the Sulbasutras along with the translation are given in [2], p.104)

ancient Sanskrit terminology – as a result, Indians could effortlessly handle large numbers right from the Vedic Age. There is terminology for all multiples of ten up to 10^{18} in early Vedic literature, the Ramayana has terms all the way up to 10^{55}, and the Jaina-Buddhist texts show frequent use of large numbers (up to 10^{140}!) for their measurements of space and time. Expressions of such large numbers are not found in the contemporary works of other nations. Even the brilliant Greeks had no terminology for denominations above the myriad (10^4) while the Roman terminology stopped with the mille (10^3). The structure of the Sanskrit numeral system and the Indian love for large numbers must have triggered the creation of the decimal system.

As we shall see later, even the smallest positive integral solution of the equation $x^2 - Dy^2 = 1$ could be very large; in fact, for $D = 61$, it is (1766319049, 226153980). The early Indian solution to this fairly deep problem could be partly attributed to the Indians' traditional fascination for large numbers and ability to play with them.

Due to the absence of good notations, the Greeks were not strong in the computational aspects of mathematics – one of the factors responsible for the eventual decline of Greek mathematics. Archimedes (287-212 BC) did realise the importance of good notation, and made notable progress to evolve one, but failed to anticipate the Indian decimal system. As the great French mathematician Laplace (1749-1827) remarked: "*The importance of this invention is more readily appreciated when one considers that it was beyond the two greatest men of antiquity: Archimedes and Apollonius.*"

"The Indians have an extremely subtle and penetrating intellect, and when it comes to arithmetic, geometry and other such advanced disciplines, other ideas must make way for theirs."

The decimal system was transmitted to Europe through the Arabs. The Sanskrit word "*sunya*" was translated into Arabic as "*sifr*" which was introduced into Germany in the 13th century as "*cifra*" from which we have the word "*cipher*". The word "*zero*" probably comes from the Latinised form "*zephirum*" of the Arabic *sifr*. Leonardo Fibonacci of Pisa (1180-1240), the first major European mathematician of the second millennium, played a major role in the spread of the Indian numeral system in Europe. The Indian notation and arithmetic eventually got standardised in Europe during the 16th-17th century.

The decimal system stimulated and accelerated trade and commerce as well as astronomy and mathematics. It is no coincidence that the mathematical and scientific renaissance began in Europe only after the Indian notation was adopted. Indeed the decimal notation is the very pillar of all modern civilization.

"I will omit all discussion of the science of the Indians, a people not the same as the Syrians; of their subtle discoveries in astronomy, discoveries that are more ingenious than those of the Greeks and the Babylonians; and of their valuable methods of calculation which surpass description." (Severus Sebokht in 662 AD)

Algebra

While sophisticated geometry was invented during the origin of the Vedic rituals, its axiomatisation and further development was done by the Greeks. The height reached by the Greeks in geometry by the time of Apollonius (260-170 BC) was not matched by any subsequent ancient or medieval civilisation. But progress in geometry proper soon reached a point of stagnation. Between the times of Pappus (300 AD) – the last big name in Greek geometry – and modern Europe, Brahmagupta's brilliant theorems (628 AD) on cyclic quadrilaterals constitute the solitary gems in the history of geometry. Further progress needed new techniques, in fact a completely new approach in mathematics. This was provided by the emergence and development of a new discipline – algebra. It is only after the establishment of an algebra culture in European mathematics during the 16th century AD that a resurgence began in geometry through its algebraisation by Descartes and Fermat in early 17th century. In fact, the assimilation and refinement of algebra had also set the stage for the remarkable strides in number theory and calculus in Europe from the 17th century.

"Indeed, if one understands by algebra the application of arithmetical operations to complex magnitudes of all sorts, whether rational or irrational numbers or space-magnitudes, then the learned Brahmins of Hindostan are the real inventors of algebra."
H Hankel

"The intellectual potentialities of the Indian nation are unlimited and not many years would perhaps be needed before India can take a worthy place in world Mathematics." (Andre Weil in 1936)

Algebra was only implicit in the mathematics of several ancient civilisations till it came out in the open with the introduction of literal or symbolic algebra in India. By the time of Aryabhata (499 AD) and Brahmagupta (628 AD), symbolic algebra had evolved in India into a distinct branch of mathematics and became one of its central pillars. After evolution through several stages, algebra has now come to play a key role in modern mathematics both as an independent area in its own right as well as an indispensable tool in other fields. In fact, the 20th century witnessed a vigorous phase of 'algebraisation of mathematics'. Algebra provides elegance, simplicity, precision, clarity and technical power in the hands of the mathematicians. It is remarkable how early the Indians had realised the signifiance of algebra and how strongly the leading Indian mathematicians like Brahmagupta (628 AD) and Bhaskara II (1150 AD) asserted and established the importance of their newly-founded discipline as we shall see in subsequent issues.

"The importance of the creation of the zero mark can never be exaggerated. This giving to airy nothing, not merely a local habitation and a name, a picture, a symbol, but helpful power, is the characteristic of the Hindu race whence it sprang. It is like coining the Nirvana into dynamos. No single mathematical creation has been more potent for the general on-go of intelligence and power."

Indians began a systematic use of symbols to denote unknown quantities and arithmetic operations. The four arithmetic operations were denoted by "yu", "ksh", "gu" and "bha" which are the first letters (or a little modification) of the corresponding Sanskrit words *yuta* (addition), *ksaya* (subtraction), *guna* (multiplication) and *bhaga* (division); similarly "ka" was used for *karani* (root), while the first letters of the names of different colours were used to denote different unknown variables. This introduction of symbolic representation was an important step in the rapid advancement of mathematics. While a rudimentary use of symbols can also be seen in the Greek texts of Diophantus, it is in India that algebraic formalism achieved full development.

The Indians classified and made a detailed study of equations (which were called *sami-karana*), introduced negative numbers together with the rules for arithmetic operations involving zero and negative numbers, discovered results on surds, described solutions of linear and quadratic equations, gave formulae for arithmetic and geometric progression as well as identitites involving summation of finite series, and applied several useful results on permutation and combinations including the formulae for ${}^nP_r$ and ${}^nC_r$. The enlargement of the number system to include negative numbers was a momentous step in the development of mathematics. Thanks to the early recognition of the existence of negative numbers, the Indians could give a unified treatment of the various forms of quadratic equations (with positive coefficients), i.e., $ax^2 + bx = c$, $ax^2 + c = bx$, $bx + c = ax^2$. The Indians were the first to recognise that a quadratic equation has two roots. Sridharacharya (750 AD) gave the well-known method of solving a quadratic equation by completing the square – an idea with far-reaching consequences in mathematics. The Pascal's triangle for quick computation of ${}^nC_r$ is described by Halayudha in the 10th century AD as *Meru-Prastara* 700 years before it was stated by Pascal; and Halayudha's Meru-Prastara was only a clarification of a rule invented by Pingala more than 1200 years earlier (around 200 BC)!

Thus, as in arithmetic, many topics in high-school algebra had been systematically developed in India. This knowledge went to Europe through the Arabs. The word *yava* in *Aryabhatiyabhasya* of Bhaskara I (6th century AD) meaning "to mix" or "to separate" has affinity with that of *al-jabr* of al-Khwarizmi (825 AD) from which the word algebra is derived. In his widely acclaimed text on history of mathematics, Cajori ([8]) concludes the chapter on India with the following remarks: "...*it is remarkable to what extent Indian mathematics enters into the science of our time. Both the form and the spirit of the arithmetic and algebra of modern times are essentially Indian. Think of our notation of numbers, brought to perfection by the Hindus, think of the Indian arithmeti-*

"The ingenious method of expressing every possible number using a set of ten symbols (each symbol having a place value and an absolute value) emerged in India. The idea seems so simple nowadays that its significance is no longer appreciated. Its simplicity lies in the way it facilitated calculations and placed arithmetic foremost among useful inventions. The importance of this invention is more readily appreciated when one considers that it was beyond the two greatest men of antiquity, Archimedes and Apollonius."

cal operations nearly as perfect as our own, think of their elegant algebraical methods, and then judge whether the Brahmins on the banks of the Ganges are not entitled to some credit."

"India has given to antiquity the earliest scientific physicians, and, according to Sir William Hunter, she has even contributed to modern medical science by the discovery of various chemicals and by teaching you how to reform misshapen ears and noses. Even more it has done in mathematics, for algebra, geometry, astronomy, and the triumph of modern science – mixed mathematics – were all invented in India, just so much as the ten numerals, the very cornerstone of all present civilization, were discovered in India, and, are in reality, sanskrit words."
Swami Vivekananda

But ancient Indian algebra went far beyond the high school level. The pinnacle of Indian achievement was attained in their solutions of the hard and subtle number-theoretic problems of finding integer solutions to equations of first and second degree. Such equations are called indeterminate or Diophantine equations. But alas, the Indian works in this area were too far ahead of the times to be noticed by contemporary and subsequent civilisations! As Cajori laments, "*Unfortunately, some of the most brilliant results in indeterminate analysis, found in the Hindu works, reached Europe too late to exert the influence they would have exerted, had they come two or three centuries earlier.*" Without some awareness of the Indian contributions in this field, it is not possible to get a true picture of the depth and skill attained in post-Vedic Indian mathematics the character of which was primarily algebraic. We shall discuss some of these number-theoretic contributions from the next instalment.

Trigonometry and Calculus

Apart from developing the subject of algebra proper, Indians also began a process of algebrisation and consequent simplification of other areas of mathematics. For instance, they developed trigonometry in a systematic manner, resembling its modern form, and imparted to it its modern algebraic character. The algebraisation of the study of infinitesimal changes led to the discovery of key principles of calculus by the time of Bhaskaracharya (1150 AD) some of which we shall mention in a subsequent issue. Calculus in India leaped to an amazing height in the analytic trigonometry of the Kerala school in the 14th century.

Although the Greeks founded trigonometry, their progress was halted due to the absence of adequate algebraic machinery and notations. Indians invented the sine and cosine functions, discovered most of the standard formulae and identitites, including the basic formula for $\sin(A \pm B)$, and constructed fairly accurate sine tables. Brahmagupta (628 AD) and Govindaswami (880 AD) gave interpolation formulae for calculating the sines of intermediate angles from sine tables – these are special cases of the Newton–Stirling and Newton–Gauss formulae for second-order difference. Remarkable approximations for π are given in Indian texts including 3.1416 of Aryabhata (499 AD), 3.14159265359 of Madhava (14th century AD) and 355/113 of Nilakanta (1500 AD). An anonymous work *Karanapaddhati* (believed to have been written by Putumana Somayajin in the 15th century AD) gives the value 3.14159265358979324 which is correct up to seventeen decimal places.

"The Hindus solved problems in interest, discount, partnership, alligation, summation of arithmetical and geometric series, and devised rules for determining the numbers of combinations and permutations. It may here be added that chess, the profoundest of all games, had its origin in India."
F Cajori

The Greeks had investigated the relationship between a chord of a circle and the angle it subtends at the centre – but their system is quite cumbersome in practice. The Indians realised the significance of the connection between a half-chord and half of the angle subtended by the full chord. In the case of a unit circle, this is precisely the sine function. The Indian half-chord was introduced in the Arab world during the 8th century AD. Europe was introduced to this fundamental notion through the work of the Arab scholar al-Battani (858-929 AD). The Arabs preferred the Indian half-chord to Ptolemy's system of chords and the algebraic approach of the Indians to the geometric approach of the Greeks.

The Sanskrit word for half-chord "*ardha-jya*", later abbreviated as "*jya*", was written by the Arabs as "*jyb*". Curiously, there is a similar-sounding Arab word "*jaib*" which means "heart, bosom, fold, bay or curve". When the Arab works were being translated into Latin, the apparently meaningless word "*jyb*" was mistaken for the

"... it is remarkable to what extent Indian mathematics enters into the science of our time. Both the form and the spirit of the arithmetic and algebra of modern times are essentially Indian. Think of our notation of numbers, brought to perfection by the Hindus, think of the Indian arithmetical operations nearly as perfect as our own, think of their elegant algebraical methods, and then judge whether the Brahmins on the banks of the Ganges are not entitled to some credit."
F Cajori

word "*jaib*" and translated as "*sinus*" which has several meanings in Latin including "heart, bosom, fold, bay or curve"! This word became "*sine*" in the English version. Aryabhata's "*kotijya*" became *cosine*.

The tradition of excellence and originality in Indian trigonometry reached a high peak in the outstanding results of Madhavacharya (1340-1425) on the power series expansions of trigonometric functions. Three centuries before Gregory (1667), Madhava had described the series

$$\theta = \tan\,\theta - (1/3)(\tan\,\theta)^3 + (1/5)(\tan\,\theta)^5 -$$
$$(1/7)(\tan\,\theta)^7 + \cdots (|\tan\,\theta| \leq 1).$$

His proof, as presented in *Yuktibhasa*, involves the idea of integration as the limit of a summation and corresponds to the modern method of expansion and term-by-term integration. A crucial step is the use of the result

$$\lim_{n\to\infty}\ (1^p + 2^p + \cdots + (n-1)^p)/n^{p+1} = 1/(p+1).$$

The explicit statement that $(|\tan\,\theta| \leq 1)$ reveals the level of sophistication in the understanding of infinite series including an awareness of convergence. Madhava also discovered the beautiful formula

$$\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \cdots,$$

obtained by putting $\theta = \pi/4$ in the Madhava–Gregory series. This series was rediscovered three centuries later by Leibniz (1674). As one of the first applications of his newly invented calculus, Leibniz was thrilled at the discovery of this series which was the first of the results giving a connection between π and unit fractions. Madhava also described the series

$$\pi/\sqrt{12} = 1 - 1/3.3 + 1/5.3^2 - 1/7.3^3 + \cdots$$

first given in Europe by A Sharp (1717). Again, three hundered years before Newton (1676 AD), Madhava had described the well-known power series expansions

$$\sin x = x - x^3/3! + x^5/5! - \cdots$$

$$\text{and} \quad \cos x = 1 - x^2/2! + x^4/4! - \cdots.$$

These series were used to construct accurate sine and cosine tables for calculations in astronomy. Madhava's values are correct, in almost all cases, to the eighth or ninth decimal place – such an accuracy was not to be achieved in Europe within three centuries. Madhava's results show that calculus and analysis had reached remarkable depth and maturity in India centuries before Newton (1642-1727) and Leibniz (1646-1716). Madhavacharya might be regarded as the first mathematician who worked in analysis!

"Incomparably greater progress than in the solution of determinate equations was made by the Hindus in the treatment of indeterminate equations. Indeterminate analysis was a subject to which the Hindu mind showed a happy adaptation."
F Cajori

Unfortunately, the original texts of several outstanding mathematicians like Sridhara, Padmanabha, Jayadeva and Madhava have not been found yet – it is only through the occasional reference to some of their results in subsequent commentaries that we get a glimpse of their work. Madhava's contributions are mentioned in several later texts including the *Tantra Samgraha* (1500) of the great astronomer Nilakanta Somayaji (1445-1545) who gave the heliocentric model before Copernicus, the *Yuktibhasa* (1540) of Jyesthadeva (1500-1610) and the anonymous *Karanapaddhati*. All these texts themselves were discovered by Charles Whish and published only in 1835.

Among ancient mathematicians whose texts have been found, special mention may be made of Aryabhata, Brahmagupta and Bhaskaracharya. All of them were eminent astronomers as well. We shall make a brief mention of some of their mathematical works in subsequent issues.

"Unfortunately, some of the most brilliant results in indeterminate analysis, found in the Hindu works, reached Europe too late to exert the influence they would have exerted, had they come two or three centuries earlier."
F Cajori

Later Developments

The Indian contributions in arithmetic, algebra and trigonometry were transmitted by the Arabs and Persians to Europe. The Arabs also preserved and transmitted the Greek heritage. After more than a thousand years of slumber, Europe rediscovered its rich Greek heritage and acquired some of the fruits of the phenomenal Indian progress. It is on the foundation formed by the blending of the two great mathematical cultures – the geometric and axiomatic tradition of the Greeks and the algebraic and computational tradition of the Indians – that the mathematical renaissance took place in Europe.

Indians made significant contributions in several frontline areas of mathematics during the 20th century, especially during the second half, although this fact is not so well-known among students partly because the frontiers of mathematics have expanded far beyond the scope of the university curricula. However, Indians virtually took no part in the rapid development of mathematics that took place during the 17th-19th century – this period coincided with the general stagnation in the national life. Thus, while high-school mathematics, especially in arithmetic and algebra, is mostly of Indian origin, one rarely comes across Indian names in college and university courses as most of that mathematics was created during the period ranging from late 17th to early 20th century. But should we forget the culture and greatness of India's millenniums because of the ignorance and weakness of a few centuries?

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Address for Correspondence
Amartya Kumar Dutta
Indian Statistical Institute
203, BT Road
Kolkata 700 032, India.

"As I look back upon the history of my country, I do not find in the whole world another country which has done quite so much for the improvement of the human mind. Therefore I have no words of condemnation for my nation. I tell them, 'You have done well; only try to do better.' Great things have been done in the past in this land, and there is both time and room for greater things to be done yet ... Our ancestors did great things in the past, but we have to grow into a fuller life and march beyond even their great achievement."

Swami Vivekananda
(Complete Works Vol III p.195)