# Exploring Spherical Geometry

## Introduction

The study of plane Euclidean geometry usually begins with segments and lines. In this investigation, you will explore analogous objects on the surface of a sphere, which has a different kind of geometry. Because we are working only on the *surface* of the sphere, this geometry is completely two-dimensional.

Geometry on the surface of a sphere is an example of an *elliptic geometry*. Because a sphere provides just one possible model of elliptic geometry, we will use the term "spherical" in the investigations that follow.

## Investigating Lines on a Sphere

These investigations require some string or rubber bands and a ball.

So that we all agree on what lines are (the shortest distance between two points), let's define lines on the sphere, or "s-lines", to be *great circles*. Although these are not of infinite length they are the best we can do. In what sense are these lines "straight?" If the radius of the sphere is 1, each s-line has a constant length. What is it?

What about length of segments on the surface of a sphere? Because the shortest distance between two points A and B is the length of the *minor arc* determined by the great circle containing A and B, we will always use minor arcs when talking about "s-segments."

Consider the following properties of lines in Neutral Geometry:

a) Two points determine a line
b) Lines contain an infinite number of points.
c) Lines have infinite length.
d) When two lines intersect, their intersection is exactly one point.
e) If A, B and C are three points on a line with A-B-C, then we cannot also have A-C-B and/or B-A-C.
f) For any real number, x, it is possible to construct a segment that has length x.
g) State at least one other property to investigate:

1. **Which properties of lines hold on a sphere?**
   It is important to remember that you are working only *on the surface* of the sphere; the lines cannot pass through the interior of the sphere. To answer parts a, b, and c below, use the ball and string/rubber bands to investigate whether the properties listed above are always true, sometimes true or never true. Be sure to justify your response with a brief explanation, examples or counterexamples.
   a) Which of the properties a-g listed above will *always* be true?

b) Which properties will *sometimes* be true?

c) Which properties will *never* be true?

2. Which of the properties a-g listed above can be modified from Neutral or Euclidean Geometry to hold true in Spherical Geometry? Name the property and then state the modified property.

3. Choosing great circles as lines means that our s-lines are “curved” and there is not as clear a distinction between line and circle as there was in the Euclidean plane. For instance, on the Euclidean plane, if a path leads back to the starting point, the path must be curved to the left or the right. But on the sphere if you follow an s-line long enough you will return to where you started. Write a definition of an s-line and a definition of an s-circle to distinguish between the two.

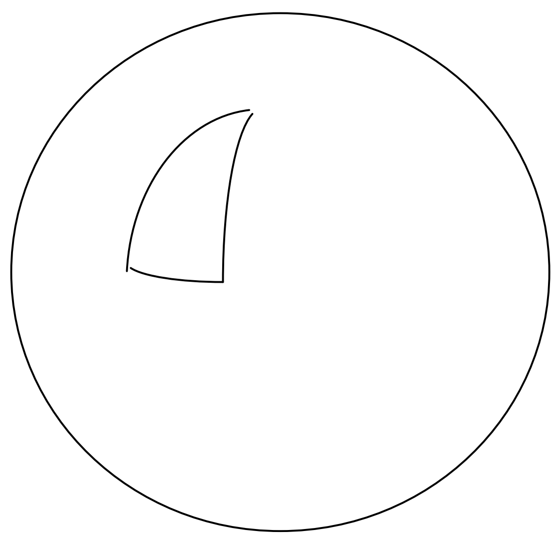
4. Find two s-lines that intersect in more than one point. How do the points of intersection relate to one another? If you choose two different lines, how will your results differ?

5. Do parallel lines exist in Spherical Geometry? Explain.

**Investigating Triangles and other Objects on a Sphere**
The drawing that follows might look odd. This is because it represents an s-triangle on the curved surface of a sphere, drawn on the flat surface of the paper. The circle represents the "edge" of the sphere when you look at it from any direction.

6. Notice that the triangle in the diagram below looks distorted.
   a) Choose three vertices of another s-triangle so that the triangle will look even more distorted if you were to draw it on a flat surface. Where would these vertices be located? Try to draw this distorted triangle on the figure below.

   b) When would a spherical triangle look least distorted in a flat drawing? Can you draw this one in the figure?



7. Based on drawings (such as those above) and experimentation (using a ball and string/rubber bands), what would you guess about the sum of the angles of a spherical triangle? Is there a range of possibilities? Explain.

8. In order to think about the answer(s) to question #7 more precisely we need to have a way of measuring angles on a sphere. Brainstorm with your group about how to measure angles on a sphere (either use the ball or the model in the figure above). Describe your method.

9. What about congruent triangles on the surface of a sphere? What congruence criteria will be necessary to show that two s-triangles are congruent? What about similar triangles?

10. Here's your chance to compare Neutral Geometry with Spherical Geometry. Fill in the chart below as best you can. Base your responses on the investigations above and your intuitive sense of geometry on the sphere. Record Neutral Geometry results or facts and corresponding Spherical Geometry facts.

| **Neutral Geometry** | **Spherical Geometry** |
|---|---|
| 1. Two points determine a line. | 1. |
| 2. Lines have infinite length. | 2. |
| 3. Angle sum of any triangle is _________. | 3. |
| 4. What do you know about exterior angles? | 4. |
| 5. How do you determine if two triangles are congruent? | 5. |
| 6. Existence and uniqueness of perpendiculars from a point to a line. | 6. |
| 7. What other results can you add to this list?<br>a)<br><br>b) | 7. |